SOLUTIONS OF NONLINEAR DIFFERENTIAL EQUATIONS
ON A RIEMANNIAN MANIFOLD
AND THEIR TRACE ON THE MARTIN BOUNDARY

E. B. DYNKIN AND S. E. KUZNETSOV

ABSTRACT. Let L be a second order elliptic differential operator on a Riemannian manifold E with no zero order terms. We say that a function h is L-harmonic if $Lh = 0$. Every positive L-harmonic function has a unique representation

$$h(x) = \int_{E'} k(x, y)\nu(dy),$$

where k is the Martin kernel, E' is the Martin boundary and ν is a finite measure on E' concentrated on the minimal part E^* of E'. We call ν the *trace of h on E'*.

Our objective is to investigate positive solutions of a nonlinear equation

(*) $$Lu = u^\alpha \quad \text{on } E$$

for $1 < \alpha \le 2$ [the restriction $\alpha \le 2$ is imposed because our main tool is the (L, α)-superdiffusion, which is not defined for $\alpha > 2$]. We associate with every solution u of (*) a pair (Γ, ν), where Γ is a closed subset of E' and ν is a Radon measure on $O = E' \setminus \Gamma$. We call (Γ, ν) the *trace of u on E'*. Γ is empty if and only if u is dominated by an L-harmonic function. We call such solutions moderate. A moderate solution is determined uniquely by its trace. In general, many solutions can have the same trace.

In an earlier paper, we investigated the case when L is a second order elliptic differential operator in $\mathbb{R}^d$ and E is a bounded smooth domain in $\mathbb{R}^d$. We obtained necessary and sufficient conditions for a pair (Γ, ν) to be a trace, and we gave a probabilistic formula for the maximal solution with a given trace.

The general theory developed in the present paper is applicable, in particular, to elliptic operators L with bounded coefficients in an arbitrary bounded domain of $\mathbb{R}^d$, assuming only that the Martin boundary and the geometric boundary coincide.

0. INTRODUCTION

0.1 Definition of the trace. The key ingredients of this definition are: (1) a concept of a moderate solution and (2) operators Q_B indexed by closed subsets of the Martin boundary E' and acting on positive solutions of the equation

(0.1) $$Lu = u^\alpha.$$

A positive solution u of (0.1) is called *moderate* if it is dominated by an L-harmonic function. For such u, there exists the minimal L-harmonic majorant h. The trace of u is defined as the finite measure ν which appears in the Martin integral representation of h.

We introduce the operators Q_B probabilistically, but they can also be defined analytically. [The equivalence of both approaches follows from [6].] If u is a solution of (0.1) and D is a regular open set in E, then there exists a minimal element v_D among positive solutions v of (0.1) in D such that, for every $c \in \partial D$, $v(x) \to u(c)$ as $x \to c$. Moreover, v_D is monotone decreasing in D. Let $d(x,y)$ be the Martin distance between x and y. For every closed subset B of E', there exists a sequence of regular open sets D_n in E with the properties that for every n, $\inf_{x \in D_n} d(x, B) > 0$ and $\sup_{x \notin D_n} d(x, B) \to 0$ as $n \to \infty$. The limit $\lim v_{D_n}$ exists, and it does not depend on the choice of D_n. We denote it by $Q_B(u)$.

The trace of a solution u can be defined as a pair (a closed set Γ, a Radon measure ν on $O = E' \setminus \Gamma$) with the following properties:

(i) For every closed $B \subset O$, $Q_B(u)$ is moderate and its trace coincides with ν on B.

(ii) If $Q_B(u)$ is moderate for $B \subset \Gamma$, then $Q_B(u) = 0$.

0.2. Traces of moderate solutions. Which measures ν are the traces of moderate solutions? The answer to this question was given in [12] in the *classical case* when L is an elliptic operator in $\mathbb{R}^d$ and E is a bounded smooth domain in $\mathbb{R}^d$. We introduced a class of exceptional sets on the boundary and we proved that a finite measure ν is the trace of a moderate solution if and only if it does not charge exceptional sets. The class of exceptional sets Γ is described by each of the following two equivalent definitions:

0.2.A. Γ has the Martin capacity 0, which means that if $\mu(\Gamma) \neq 0$, then

$$\int_E \gamma(dx)[\int_\Gamma k(x,y)\mu(dy)]^\alpha = \infty.$$

(Here $\gamma(dx) = g(c,x)dx$, $g(x,y)$ is the Green's function, c is a reference point and $k(x,y)$ is the Martin kernel.)

0.2.B. Γ is $\mathcal{R}$-polar, i.e. it is not hit by the range $\mathcal{R}$ of (L, α)-superdiffusion (see definitions in Sections 1.2 and 1.3).

In the general case that we are studying now, the equivalence of definitions 0.2.A and 0.2.B is not proved. However we established in [8] that 0.2.B implies 0.2.A. It is natural to call sets which satisfy 0.2.A *weakly exceptional* and sets which satisfy 0.2.B *strongly exceptional*. We proved in [8] that the condition "ν does not charge weakly exceptional sets" is sufficient and the condition "ν does not charge strongly exceptional sets" is necessary for ν to be the trace of a moderate solution u.

0.3. Traces of arbitrary solutions. In Sections 6 and 7 we investigate the traces (Γ, ν) of arbitrary solutions of (0.1) under the assumption

(N) $Q_B[Q_\Gamma(u)] = 0$ for all solutions u and all pairs of disjoint closed sets B, Γ.

In the classical case, (N) holds by Theorem 3.1 in [12]. In Section 8 we establish more general conditions sufficient for (N). [At present we have no examples when (N) is false.]

The following two theorem are proved under assumption (N).

Theorem A. *The trace* (Γ, ν) *of every solution* u *has the following properties:*

0.3.A. ν *does not charge strongly exceptional sets.*

0.3.B. Every nonempty relatively open strongly exceptional subset B of Γ contains explosion points of ν.

[We say that c is an explosion point of ν if $\nu(U) = \infty$ for every neighborhood U of c.]

Theorem B. *Suppose that* (Γ, ν) *satisfies 0.3.B and the following condition:*

0.3.A*. ν *does not charge weakly exceptional sets.*

Then (Γ, ν) *is the trace of a solution.*

0.4. On assumption $(\mathbb{N})$**.** In Section 8 we introduce the concept of a normal point at the Martin boundary E', and we prove that assumption $(\mathbb{N})$ is satisfied if all points are normal. The definition of a normal point is probabilistic, but we also give an analytic criterion of normality (the existence of an appropriate barriers).

The general theory is then applied to the case when E is a bounded domain in $\mathbb{R}^d$ and L is an elliptic operator in E. We show that $(\mathbb{N})$ holds if:

(a) the coefficients of L are bounded;

(b) the Martin boundary E' coincides with the boundary ∂E of E in Euclidean geometry.

In fact, (a) can be replaced by a weaker condition 8.5.B and, instead of (b), it is sufficient to assume that one of the following two conditions is satisfied:

(a') The transition from ∂E to E' does not lead to "pasting together" any pair of points.

(a") No "pasting" is needed for transition from E' to ∂E, and only a finite number of points of E' correspond to each point of ∂E.

[More precisely, put $c \hookrightarrow b$ if $c \in E'$, $b \in \partial E$ and if there exists a sequence $x_n \in E$ such that $x_n \to c$ in the Martin topology and $x_n \to b$ in the Euclidean topology. Condition (a') means that $c \hookrightarrow b, c' \hookrightarrow b$ implies $c = c'$. Condition (a") means that $c \hookrightarrow b, c \hookrightarrow b'$ implies $b = b'$ and, for every b, there exist only a finite number of c such that $c \hookrightarrow b$.]

One of conditions (a'), (a") holds in all examples described in [1] to demonstrate distinctions between two boundaries.

0.5. Probabilistic solution. The solution described in Theorem B is defined by the formula

$$(0.2) \qquad\qquad u(x) = -\log P_x\{\mathcal{R} \cap \Gamma = \emptyset, e^{-A_\nu}\}$$

where $\mathcal{R}$ is the range of (L, α)-superdiffusion (X, P_x) and A_ν is a continuous linear additive functional of (X, P_x) with spectral measure ν (see definitions in Sections 1.2, 1.3 and 4.3). In Section 7 we prove that the solution (0.2) dominates all solutions with the same trace.

In general many solutions with the same trace can be constructed by applying formula (0.2) to various Borel sets $\Gamma \subset E'$ with the same closure. [An example of nonuniqueness based on this idea was communicated to us by J.-F. Le Gall.]

0.6. Bibliographical notes. Open problems. A pair (Γ, ν) as a characteristic of a positive solution of the equation $\Delta u = u^\alpha$ first appeared in Le Gall's paper [13]. He proved that in the case of the unit disk D in the plane and $\alpha = 2$, there is a 1-1 correspondence between all positive solutions and all pairs (Γ, ν), where Γ is a closed subset of ∂D and ν is a Radon measure on $\partial D \setminus \Gamma$ (in this

case, there exist no exceptional sets, and therefore conditions 0.3.A, 0.3.A' and 0.3.B hold automatically). For constructing a solution, Le Gall uses the Brownian snake — a path-valued Markov process that he discovered, closely related to $(\Delta, 2)$-superdiffusion. In [14] he extended the result to arbitrary planar domains of class C^2.

Marcus and Véron have announced in [15] and [16] and proved in [17] and [17a] results on the equation $\Delta u = u^\alpha$ in the unit ball of $\mathbb{R}^d$ for arbitrary $\alpha > 1$ and d. [The name "trace" for a pair (Γ, ν) was first suggested in those publications.] By purely analytic means, they proved the existence and uniqueness of a solution with a given trace in the case $\alpha < (d+1)/(d-1)$ (where there exist no exceptional sets). In the case $\alpha \geq (d+1)/(d-1)$, they proved the existence of a solution under a condition on ν which is stronger than 0.3.A*.

The definition of the trace in [13], [14], [15] and [16] is different from ours. It is based on asymptotic behavior of a solution near the boundary.

The equation $Lu = u^\alpha$ for an elliptic operator L in $\mathbb{R}^d$ and a bounded domain of class $C^{2,\lambda}$ has been investigated in [12]. Necessary and sufficient conditions have been established for a pair (Γ, ν) to be a trace. (Note that in this case conditions 0.3.A and 0.3.A* are equivalent, and therefore the results in [12] follow from Theorems A and B.)

An important role was played in [12] by the characterization of $Q_B(u)$ as the maximal solution dominated by u and equal to 0 on $\partial E \setminus \mathcal{B}$. (Condition $(\mathbb{N})$ is a direct implication of this fact.) Such a simple characterization is not available in the general setting. Among tools which we use to overcome this difficulty is a stochastic version of the comparison principle (see Theorem 2.1).

Several challenging problems remain open:

A. Do there exist any positive solutions of (0.1) which cannot be represented in the form (0.2) with a suitable coanalytic set Γ and measure ν?

B. Is it true in the general setting that the trace uniquely determines a solution if there are no exceptional sets on E'?

C. Can the class of weakly exceptional sets be wider than the class of strongly exceptional sets?

Acknowledgments

1. Diffusions and superdiffusions

1.1. Diffusions. A general second order elliptic differential operator with no zero order terms on a Riemannian manifold E has the form $L = \Delta + Y$, where Δ is the Laplace-Beltrami operator and Y is a first order differential operator (i.e., a vector field). We denote by $m(dx)$ the measure on E determined by the Riemannian metric. The L-diffusion is a Markov process $\xi = (\xi_t, \Pi_x)$ on E with continuous paths and with the transition function $p_t(x, y)m(dy)$ where $p_t(x, y)$ satisfies the following conditions:

1.1.A. For all t, x,

$$\int_E p_t(x, y)m(dy) \leq 1.$$

1.1.B. For all $s, t > 0$ and all x, z,

$$\int_E p_s(x, y)m(dy)p_t(y, z) = p_{s+t}(x, z).$$

1.1.C. For every bounded positive function $\varphi \in C^1(E)$,

$$v_t(x) = \int_E p_t(x, y)\varphi(y)m(dy)$$

is a solution of the boundary value problem

$$\frac{\partial v}{\partial t} = Lv \quad \text{for } t > 0,$$
$$v \to \varphi \quad \text{as } t \downarrow 0.$$

The existence of a function $p_t(x, y)$ subject to conditions 1.1.A-C has been proved in [19], and the existence of the corresponding L-diffusion follows from [2].

If $L = \Delta$, then the corresponding L-diffusion is called the Brownian motion on E.

Solutions of the equation $\Delta h = 0$ are called harmonic functions. We use the name L-harmonic functions for solutions of the equation $Lh = 0$. Every positive L-harmonic function h has a unique representation

(1.1) $$h(x) = \int_{E'} k(x, y)\nu(dy),$$

where k is the Martin kernel and ν is a finite measure on the Martin boundary E' concentrated on the minimal part E^* of E'. We call ν the *trace of h*.

Recall that the Martin metric $d(x, y)$ in E is defined in terms of k, that the *exit space* $\hat{E}$ is a compact metric space obtained by completion of E with respect to d and that $E' = \hat{E} \setminus E$. We say M-limit, M-closure etc. when referring to the corresponding topology.

A path ξ_t is defined on a random time interval $[0, \zeta)$. The M-limit

(1.2) $$\xi_{\zeta-} = \lim_{t \uparrow \zeta} \xi_t$$

exists, a.s., [1] and it belongs to E^*. Let σ be the probability measure on E' which appears in the Martin representation (1.1) of the L-harmonic function $h = 1$. Then for every positive Borel function φ and for all x,

(1.3) $$\Pi_x\varphi(\xi_{\zeta-}) = \int k(x, y)\varphi(y)\sigma(dy).$$

[In other words, $\varphi(y)\sigma(dy)$ is the trace of $h(x) = \Pi_x\varphi(\xi_{\zeta-})$.] More generally, to every L-harmonic function h there corresponds a measure Π_x^h on the path space such that, for every $B \subset E'$,

(1.4) $$\Pi_x^h\{\xi_{\zeta-} \in B\} = \int_B k(x, y)\nu(dy)$$

(ν is the trace of h). For every stopping time τ and every positive Borel function φ,

(1.5) $$\Pi_x^h\{\varphi(\xi_\tau)1_{\tau<\zeta}\} = \Pi_x(\varphi h)(\xi_\tau).$$

---

[1] Writing "a.s." means Π_x-a.s. for all $x \in E$.

Moreover,

$$(1.6) \qquad \Pi_x^h = \int \Pi_x^y \nu(dy),$$

where Π_x^y is the measure corresponding to the L-harmonic function $k(\cdot, y)$. [The measure $\frac{1}{h(x)} \Pi_x^h$ is called the h-transform of Π_x.]

We assume that the function

$$g(x, y) = \int_0^\infty p_t(x, y) dt$$

is finite[2] for all $x \neq y$, and we call it Green's function. Green's operator acts on positive Borel functions by the formula

$$(1.7) \qquad G\varphi(x) = \int_E g(x, y)\varphi(y)m(dy).$$

Every domain D in E is a submanifold of E. The L-diffusion on D can be obtained by killing the path at the first exit time $\tau = \inf\{t : \xi_t \notin D\}$ from D. We denote by g_D and G_D the corresponding Green's function and Green's operator.

We write ∂D for the boundary of D in the topology of E. We put $\bar{D} = D \cup \partial D$.

We say that D_n is a *standard sequence approximating* D if D_n are open subsets of D with compact closures $\bar{D}_n$ and if $\bar{D}_n \uparrow D$.

1.2. Superdiffusions. This is a mathematical model of a random cloud. The spatial motion of its infinitesimal parts is described by the L-diffusion $\xi = (\xi_t, \Pi_x)$, and the branching mechanism is determined by a parameter $\alpha \in (1, 2]$. To every open set D in E and to every $\mu \in \mathcal{M} = \mathcal{M}(E)$ [3] there corresponds a random measure (X_D, P_μ), called the *exit measure from* D. X_D describes the mass distribution of the cloud instantaneously frozen on $E \setminus D$, and P_μ is a probability measure corresponding to the initial mass distribution μ. All P_μ have the same domain $\mathcal{F}$. For every positive Borel function f,

$$(1.8) \qquad P_\mu \exp\langle -f, X_D \rangle = \exp\langle -u, \mu \rangle,$$

where

$$(1.9) \qquad u + \mathcal{E}_D(u) = K_D f$$

with

$$(1.10) \qquad \mathcal{E}_D(u)(x) = G_D(u^\alpha)(x),$$

$$(1.11) \qquad K_D f(x) = \Pi_x f(\xi_\tau)$$

(τ is the first exit time of ξ from D). [4]

We denote by P_x the measure P_{δ_x} corresponding to Dirac's measure at the point x.

The joint probability distribution of $X_{D_1}, \ldots, X_{D_n}$ is determined by (1.9) and by the Markov property: for every positive $\mathcal{F}_{\supset D}$-measurable Y,

$$(1.12) \qquad P_\mu \{ Y | \mathcal{F}_{\subset D} \} = P_{X_D} Y,$$

where $\mathcal{F}_{\subset D}$ is the σ-algebra generated by X_{D_1} with $D_1 \subset D$ and $\mathcal{F}_{\supset D}$ is the σ-algebra generated by X_{D_2} with $D_2 \supset D$.

---

[2] Otherwise $g(x, y) = \infty$ for all x, y and (*) has no positive solutions.

[3] We denote by $\mathcal{M}(S)$ the space of all finite measures on a measurable space S.

[4] The right side in (1.11) means $\int_{\tau < \zeta} f(\xi_\tau) d\Pi_x$.

The existence of a family (X_D, P_μ) subject to conditions (1.9) and (1.12) is proved in [3].

It follows from (1.8)–(1.11) that

$$(1.13) \qquad P_\mu \langle f, X_D \rangle = \langle K_D f, \mu \rangle.$$

1.3. Range and polar sets. Consider the class $\mathbb{C}$ of all closed random sets $C(\omega)$ with the property that every exit measure X_D is concentrated, a.s., on C. There exists a minimal element of $\mathbb{C}$, and it is defined uniquely up to indistinguishability. We denote it by $\mathcal{R}$ and call it the range of X. A set $B \subset E'$ is called $\mathcal{R}$-polar if

$$(1.14) \qquad P_x \{ \mathcal{R} \cap B = \emptyset \} = 1 \qquad \text{for all } x \in D.$$

1.4. Markov process (X_t, P_μ). Besides exit measures X_D we also consider random measures X_t which describe the mass distribution at a fixed time t. For every positive Borel function f,

$$(1.15) \qquad P_\mu \exp \langle -f, X_t \rangle = \exp \langle -v_t, \mu \rangle,$$

where

$$(1.16) \qquad v_t(x) + \Pi_x \int_0^t v_{t-s}^\alpha(\xi_s) \, ds = \Pi_x f(\xi_t).$$

Random measures (X_t, P_μ) form a Markov process in the state space $\mathcal{M}$. This is a more traditional model of superdiffusion than the model described in Section 1.2.

2. Equation $Lu = u^\alpha$. Operators V_D

2.1. Class $\mathcal{U}(D)$. We denote by $\mathcal{U}(D)$ the class of all positive u such that

$$(2.1) \qquad Lu = u^\alpha \quad \text{in } D,$$

and we put $\mathcal{U} = \mathcal{U}(E)$. We have:

2.1.A. If $u + \mathcal{E}_D(u)$ is L-harmonic in D, then $u \in \mathcal{U}(D)$.

2.1.B. For every positive Borel f,

$$u(x) = -\log P_x e^{-\langle f, X_D \rangle}$$

belongs to $\mathcal{U}(D)$.

2.1.C. If $u_n \in \mathcal{U}(D)$ converge pointwise on D to u, then $u \in \mathcal{U}(D)$.

2.1.D. (The mean value property) If $u \in \mathcal{U}$ and if $\bar{D}$ is compact, then

$$u(x) = -\log P_x e^{-\langle u, X_D \rangle} \quad \text{in } D.$$

2.1.A follows from Theorem 1.3 in [8]. By (1.8)–(1.9), 2.1.B follows from 2.1.A. Property 2.1.C is proved in [4, Theorem 1.2] and 2.1.D follows from Lemma 1.1 in [4] or Theorem 2.3 in [7].

2.2. Comparison principle. Let τ be the first exit time from an open set D. For every real-valued function f on D, we set

$$(2.2) \qquad f^D = \limsup_{t \uparrow \tau} f(\xi_t) \quad \text{on } \{\tau > 0\}.$$

In particular,

$$(2.3) \qquad f^E = \limsup_{t \uparrow \zeta} f(\xi_t).$$

If $B \subset \partial D$ and if f is a continuous function on $D \cup B$, then

$$(2.4) \qquad f^D = f(\xi_\tau) \quad \text{on } \{\xi_\tau \in B, \tau > 0\}.$$

Lemma 2.1. *Suppose that φ is a bounded function on ∂D and*

$$(2.5) \qquad f(x) = \Pi_x \varphi(\xi_\tau).$$

Then

$$(2.6) \qquad f^D = \lim_{t \uparrow \tau} f(\xi_t) = \begin{cases} \varphi(\xi_\tau) & \text{a.s. on } \{0 < \tau < \zeta\}, \\ 0 & \text{a.s. on } \{\tau = \zeta\}. \end{cases}$$

If $\varphi \geq 0$ and

$$(2.7) \qquad f = G_D\varphi < \infty \quad \text{on } D,$$

then

$$(2.8) \qquad f^D = 0 \quad \text{a.s. on } \{\tau > 0\}.$$

Proof. If f is given by (2.5), then $f(\xi_{t \wedge \tau})$ is a bounded continuous martingale, and therefore $\lim_{t \uparrow \tau} f(\xi_t)$ exists a.s. Clearly, it is equal to f^D. Let τ_n be the first exit times from a standard sequence D_n approximating D. Then, a.s. on $\{0 < \tau < \zeta\}$, $\tau_n < \tau$ and $\tau_n \to \tau$. Therefore $f^D = \lim f(\xi_{\tau_n})$. On the other hand,

$$(2.9) \qquad f(\xi_{\tau_n}) = \Pi_x\{\varphi(\xi_\tau)|\mathcal{F}_{\tau_n}\} \to \varphi(\xi_\tau) \quad \text{a.s. on } \{0 < \tau < \zeta\}$$

because ξ_τ is measurable with respect to $\bigvee \mathcal{F}_{\xi_{\tau_n}}$. Besides,

$$(2.10) \qquad \Pi_{\xi_{\tau_n}}\{\tau < \zeta\} = \Pi_x\{\tau < \zeta|\mathcal{F}_{\tau_n}\} \to 0 \quad \text{a.s. on } \{\tau = \zeta\}.$$

Formula (2.6) follows from (2.9) and (2.10). If f is given by (2.7), then $f(\xi_{\tau \wedge t})$ is a positive supermartingale and

$$f^D = \lim_{t \uparrow \tau} f(\xi_t) = \lim_{t \uparrow \tau} \Pi_{\xi_t} \int_0^\tau \varphi(\xi_s)ds.$$

Formula (2.8) follows from the relation

$$\Pi_x \Pi_{\xi_{\tau_n}} \int_0^\tau \varphi(\xi_s)ds = \Pi_x \int_{\tau_n}^\tau \varphi(\xi_s)ds \to 0 \quad \text{a.s. on } \{0 < \tau\}.$$

Lemma 2.2. *Suppose that $u \in C^2(D)$ is bounded above and*

$$(2.11) \qquad Lu \geq 0 \quad \text{on } D.$$

If

$$(2.12) \qquad u^D \leq 0 \quad \Pi_x\text{-a.s.}$$

for all $x \in D$, then $u \leq 0$ in D.

Proof. Let D_n be a standard sequence approximating D and let τ_n be the first exit time from D_n. By Ito's formula,

$$M_t = u(\xi_{\tau_n \wedge t}) - \int_0^{\tau_n \wedge t} Lu(\xi_s)ds$$

is a martingale relative to all measures Π_x. Therefore $\Pi_x M_t = \Pi_x M_0 = u(x)$, and (2.11) implies $\Pi_x u(\xi_{\tau_n \wedge t}) \geq u(x)$. Since $\tau_n \uparrow \tau$ a.s., Fatou's lemma implies $\Pi_x u(\xi_{\tau \wedge t}) \geq u(x)$. By applying Fatou's lemma once more, we conclude that $\Pi_x u^D \geq u(x)$ and $u \leq 0$ by (2.12). $\qquad \square$

Theorem 2.1 (Comparison principle). *Suppose $u, v \in C^2(D)$ satisfy the following conditions:*

$$(2.13) \qquad\qquad Lu - u^\alpha \leq Lv - v^\alpha \quad \text{in } D,$$

$v - u$ is bounded above and

$$(2.14) \qquad\qquad (v - u)^D \leq 0 \quad \Pi_x\text{-a.s.}$$

for all $x \in D$. Then $v \leq u$ in D.

Proof. Put $w = v - u$. Suppose $\tilde{D} = \{w > 0\}$ is nonempty. Note that $Lw = Lv - Lu \geq v^\alpha - u^\alpha > 0$ on $\tilde{D}$. Let τ and $\tilde{\tau}$ be the first exit times from D and $\tilde{D}$. By (2.4), $w^{\tilde{D}} = w(\xi_{\tilde{\tau}}) = 0$ a.s. on $\{\tilde{\tau} < \tau\}$. By (2.14), $w^D = 0$ a.s. on $\{\tilde{\tau} = \tau\}$. It follows from Lemma 2.2 (applied to $\tilde{D}$) that $w \leq 0$ on $\tilde{D}$, which contradicts the definition of $\tilde{D}$. $\qquad\square$

2.3. Operators V_D. With every open set D we associate an operator which acts on positive Borel functions on E by the formula

$$(2.15) \qquad\qquad V_D(f)(x) = -\log P_x e^{-\langle f, X_D \rangle}.$$

By 2.1.B, $V_D(f) \in \mathcal{U}(D)$. If $x \notin D$, then $P_x\{X_D = \delta_x\} = 1$, and therefore $V_D(f) = f$ on $E \setminus D$. It follows from (1.8)–(1.9) that

$$(2.16) \qquad\qquad V_D(f) \leq K_D f.$$

The following properties of the operators V_D are proved in [12, Section 2.2]:

2.3.A. $V_D(f_1) \leq V_D(f_2)$ for $f_1 \leq f_2$.

2.3.B. If $D_1 \subset D_2$, then $V_{D_1} V_{D_2} = V_{D_2} = V_{D_2} V_{D_1}$.

2.3.C. For every f_1, f_2,

$$V_D(f_1 + f_2) \leq V_D(f_1) + V_D(f_2).$$

2.3.D. For every D_1, D_2,

$$V_{D_1 \cap D_2}(f) \leq V_{D_1}(f) + V_{D_2}(f) \quad \text{in } D_1 \cap D_2.$$

2.4. Action of V_D on solutions.

Theorem 2.2. *If $u \in \mathcal{U}$, then*

$$(2.17) \qquad\qquad V_D(u) \leq u.$$

If $D_1 \subset D_2$, then

$$(2.18) \qquad\qquad V_{D_2}(u) \leq V_{D_1}(u).$$

Proof. Put $u_B = u 1_B$. First, we prove that, for every compact set $B \subset \partial D$, the function $w = V_D(u_B)$ has the properties

$$(2.19) \qquad\qquad w^D = \begin{cases} u_B(\xi_\tau) & \text{a.s. on } \{0 < \tau < \zeta\}, \\ 0 & \text{a.s. on } \{\tau = \zeta\}. \end{cases}$$

and

$$(2.20) \qquad\qquad w \leq u.$$

By (1.8)–(1.9), we have

$$(2.21) \qquad\qquad w + \mathcal{E}_D(w) = h \quad \text{in } D$$

with $h(x) = \Pi_x u_B(\xi_\tau)$. By Lemma 2.1,

$$h^D = \begin{cases} u_B(\xi_\tau) & \text{a.s. on } \{0 < \tau < \zeta\}, \\ 0 & \text{a.s. on } \{\tau = \zeta\}; \end{cases}$$

$$[\mathcal{E}_D(w)]^D = 0 \quad \text{a.s. on } \{\tau > 0\}$$

and (2.21) implies (2.19). By (2.4) and (2.19), $u^D = u(\xi_\tau) \geq u^B(\xi_\tau) = w^D$ a.s. on $\{0 < \tau < \zeta\}$ and $u^D \geq 0 = w^D$ a.s. on $\{\tau = \zeta\}$. Hence, $(w - u)^D \leq 0$ a.s.

Since w is bounded and $Lu - u^\alpha = Lw - w^\alpha = 0$ in D, (2.20) follows from Theorem 2.1.

Consider compact sets B_n such that $B_n \uparrow \partial D$, and let $w_n = V_D(u_{B_n})$. Clearly, $w_n \uparrow V_D(u)$, and (2.20) implies (2.17).

If $D_1 \subset D_2$, then, by 2.3.B, (2.17) and 2.3.A, $V_{D_2}(u) = V_{D_1}V_{D_2}(u) \leq V_{D_1}(u)$. $\square$

Remark. For every $c \geq 0$,

(2.22) $$V_D(cu) \leq (c \vee 1)u.$$

If $c < 1$, this follows from 2.3.A. If $c \geq 1$, then $L(cu) - (cu)^\alpha \leq 0$, and we get (2.22) by the arguments used in proof of (2.17) with $w = V_D(cu_B)$.

3. Probabilistic representation of solutions

3.1. Stochastic boundary value of a solution u. Denote by $\mathcal{O}$ the class of all open sets D in E such that $\bar{D}$ is compact. Recall that according to Section 1.2, $\mathcal{F}_{\supset D}$ stands for the σ-algebra generated by $X_{\tilde{D}}$ with $\tilde{D} \supset D$. We use the name the *germ σ-algebra* of X on the Martin boundary for the intersection $\mathcal{F}(E')$ of $\mathcal{F}_{\supset D}$ over all $D \in \mathcal{O}$. Note that

(3.1) $$\mathcal{F}(E') = \bigcap_n \mathcal{F}_{\supset D_n}$$

for every standard sequence D_n approximating E.

Theorem 3.1. *For every $u \in \mathcal{U}$, there exists a $\mathcal{F}(E')$-measurable function $Z \geq 0$ such that*

(3.2) $$\lim \langle u, X_{D_n} \rangle = Z \quad P_\mu\text{-a.s.}$$

for every standard sequence D_n approximating E and every $\mu \in \mathcal{M}$. For all $\mu \in \mathcal{M}$,

(3.3) $$P_\mu e^{-Z} = e^{-\langle u, \mu \rangle}.$$

In particular,

(3.4) $$u(x) = -\log P_x e^{-Z} \qquad \text{for all } x \in E.$$

Proof. Suppose that D_n is a standard sequence approximating E and put $Y_n = e^{-\langle u, X_{D_n} \rangle}$. By (1.12), (1.8), (2.15) and (2.17),

$$P_\mu\{Y_{n+1}|\mathcal{F}_{\subset D_n}\} = P_{X_{D_n}}Y_{n+1} = e^{-\langle V_{D_{n+1}}(u), X_{D_n} \rangle} \geq Y_n.$$

Hence $(Y_n, \mathcal{F}_{\subset D_n}, P_\mu)$ is a bounded submartingale, which implies the existence, P_μ-a.s., of $\lim \langle u, X_{D_n} \rangle$.

Fix a standard sequence D_n^0 approximating E and put $Z = \limsup \langle u, X_{D_n^0} \rangle$. If D_n is an arbitrary standard sequence, then there exist $n_1 < n_2 < \cdots$ such that

(3.5) $$D_{n_1}^0 \subset D_{n_2} \subset D_{n_3}^0 \subset D_{n_4} \subset \cdots.$$

There exists, P_μ-a.s., a limit of $\langle u, X_D \rangle$ along the standard sequence (3.5). It coincides, P_μ-a.s., with the limit along D_n and the limit along D_n^0, which proves (3.2). Formula (3.3) follows from (3.2) and the dominated convergence theorem. $\square$

We call Z defined in Theorem 3.1 the *stochastic boundary value of u*. (If two functions Z_1 and Z_2 satisfy this definition, then $Z_1 = Z_2$ P_μ-a.s. for all $\mu \in \mathcal{M}$.) The function u defined by formula (3.4) is called the *log-potential of Z*. Theorem 3.1 can be restated as follows:

Theorem 3.1 bis. *For every $u \in \mathcal{U}$, there exists a stochastic boundary value Z. The log-potential of Z is equal to u.*

3.2. Total classes of measures. We say that a subset $\mathcal{M}^*$ of a set $\mathcal{M}$ is *total* if:
   3.2.A. If $\mu \in \mathcal{M}^*$ and if $\tilde{\mu} \leq \mu$, then $\tilde{\mu} \in \mathcal{M}^*$.
   3.2.B. For every $\mu \in \mathcal{M}^*$ and for an arbitrary $D \in \mathcal{O}$, $P_\mu\{X_D \in \mathcal{M}^*\} = 1$.
   3.2.C. $\mathcal{M}^*$ contains all Dirac's measures $\delta_x, x \in E$.
   Clearly, the intersection of any countable family of total sets is a total set.
   To an arbitrary L-harmonic function h there corresponds a total set $\mathcal{M}^*(h) = \{\mu \in \mathcal{M} : \langle h, \mu \rangle < \infty\}$. The class $\mathcal{M}_{\mathcal{E}}^*(h) = \langle h + \mathcal{E}(h), \mu \rangle < \infty$ satisfies 3.2.A, B. If $\mathcal{E}(h)(x) < \infty$ for some x, then $\mathcal{E}(h)(x) < \infty$ for all x (this follows from Harnack's inequality). Hence, $\mathcal{M}_{\mathcal{E}}^*(h)$ satisfies 3.2.C and is a total class.
   One more example of a total set: put $\mu \in \mathcal{M}_0^*$ if $\mu \in \mathcal{M}$ and if $\mu(E \setminus D) = 0$ for some $D \in \mathcal{O}$. Clearly, $\mathcal{M}_0^*$ satisfies 3.2.A and C. By [5, I.9.A], it satisfies 3.2.B.
   Two functions Z_1 and Z_2 are called $\mathcal{M}^*$-*equivalent* if $P_\mu\{Z_1 \neq Z_2\} = 0$ for all $\mu \in \mathcal{M}^*$.

3.3. Class $\mathfrak{Z}(\mathcal{M}^*)$. Let $\mathcal{M}^*$ be a total set. Denote by $\mathcal{F}(E', \mathcal{M}^*)$ the completion of $\mathcal{F}(E')$ with respect to all $P_\mu, \mu \in \mathcal{M}^*$. The class $\mathfrak{Z}(\mathcal{M}^*)$ consists of all positive $\mathcal{F}(E', \mathcal{M}^*)$-measurable functions Z subject to the conditions:
   CB$(\mathcal{M}^*)$. [5] For every $\mu \in \mathcal{M}^*$,

$$(3.6) \qquad -\log P_\mu e^{-Z} = \int [-\log P_x e^{-Z}]\mu(dx).$$

and
   FIN. For all $x \in E$, $P_x\{Z < \infty\} > 0$.
   We use the shorter notation $\mathfrak{Z}$ for $\mathfrak{Z}(\mathcal{M})$.

Theorem 3.2. *A stochastic boundary value Z of any $u \in \mathcal{U}$ belongs to $\mathfrak{Z}$. Let $Z \in \mathfrak{Z}(\mathcal{M}^*)$. Then the log-potential u of Z belongs to $\mathcal{U}$, and Z is $\mathcal{M}^*$-equivalent to a stochastic boundary value of u.*

Proof. 1°. The stochastic boundary value is $\mathcal{F}(E')$-measurable. Property CB$(\mathcal{M})$ follows from (3.3) and (3.4). Since $u(x) < \infty$ for all $x \in E$, (3.4) implies FIN.
   2°. Let u be the log-potential of $Z \in \mathfrak{Z}(\mathcal{M}^*)$. By FIN, $u < \infty$. Suppose $\mu \in \mathcal{M}^*$. By the Markov property (1.12),

$$(3.7) \qquad P_\mu\{e^{-Z}|\mathcal{F}_{\subset D}\} = P_{X_D}e^{-Z} \quad P_\mu\text{-a.s.}$$

for every $D \in \mathcal{O}$. Then, by CB$(\mathcal{M}^*)$ and 3.2.B,

$$(3.8) \qquad P_{X_D}e^{-Z} = e^{-\langle u, X_D \rangle} \quad P_\mu\text{-a.s.}$$

---

[5] CB stands for "continuous branching".

Let D_n be a standard sequence approximating E. All exit measures $X_D, D \in \mathcal{O}$, are measurable relative to the σ-algebra $\bigvee_n \mathcal{F}_{\subset D_n}$. Therefore Z is also measurable with respect to this σ-algebra and, by (3.1), (3.7) and (3.8),

$$e^{-Z} = \lim P_\mu \{ e^{-Z} | \mathcal{F}_{\subset D_n} \} = \lim e^{-\langle u, X_{D_n} \rangle} \quad P_\mu\text{-a.s.,}$$

which implies

$$(3.9) \qquad \lim \langle u, X_{D_n} \rangle = Z \quad P_\mu\text{-a.s.}$$

It follows from (3.4) and (3.9) that

$$u(x) = \lim u_n(x),$$

where

$$u_n = V_{D_n}(u) \in \mathcal{U}(D_n).$$

By 2.1.C, $u \in \mathcal{U}$. By (3.9), Z is $\mathcal{M}^*$-equivalent to a stochastic boundary value of u determined by (3.2). $\qquad \square$

It follows from Theorem 3.2 that, for every $Z \in \mathfrak{Z}(\mathcal{M}^*)$, there exists an $\mathcal{M}^*$-equivalent function $\tilde{Z} \in \mathfrak{Z}$.

We denote by $\bar{\mathfrak{Z}}(\mathcal{M}^*)$ the set obtained from $\mathfrak{Z}(M^*)$ by identifying $\mathcal{M}^*$-equivalent functions. Theorem 3.2 implies:

Theorem 3.2 bis. *Formula (3.4) defines a 1-1 mapping from $\bar{\mathfrak{Z}}(\mathcal{M}^*)$ onto $\mathcal{U}$. The inverse mapping is given by (3.9).*

3.4. Properties of log-potential.

Lemma 3.1. *Suppose $Z_1, Z_2 \in \mathfrak{Z}(\mathcal{M}^*)$ and let u_1, u_2, u be the log-potentials of Z_1, Z_2 and $Z = Z_1 + Z_2$. Then*

$$(3.10) \qquad u \le u_1 + u_2.$$

Proof. Proposition 2.3.C and Theorem 2.2 imply that if $u_1, u_2 \in \mathcal{U}$, then, for every D,

$$(3.11) \qquad V_D(u_1 + u_2) \le u_1 + u_2.$$

Let $x \in E$. By (3.2),

$$(3.12) \qquad Z = \lim \langle u_1 + u_2, X_{D_n} \rangle \quad P_x\text{-a.s.}$$

By (3.12), (2.15) and by the dominated convergence theorem,

$$(3.13) \qquad u(x) = \lim V_{D_n}(u_1 + u_2)(x).$$

Formula (3.10) follows from (3.11) and (3.13). $\qquad \square$

Theorem 3.3. *If $\mathcal{M}^* \subset \mathcal{M}_0^*$, then $\mathfrak{Z}(\mathcal{M}^*)$ is a convex cone. The log-potential is a subadditive function on this cone.*

Proof. Suppose $Z_1, Z_2 \in \mathfrak{Z}(\mathcal{M}^*)$ and let $Z = c_1 Z_1 + c_2 Z_2$, where c_1, c_2 are positive constants. Denote by u_1, u_2, u the log-potentials of Z_1, Z_2, Z. Let $\mu \in \mathcal{M}^*$. It follows from (3.2) that

$$(3.14) \qquad Z = \lim \langle c_1 u_1 + c_2 u_2, X_{D_n} \rangle \quad P_\mu\text{-a.s.,}$$

which implies

$$(3.15) \qquad -\log P_\mu e^{-Z} = \lim \langle V_{D_n}(c_1 u_1 + c_2 u_2), \mu \rangle.$$

Formula (3.15) applied to δ_x yields

$$(3.16) \qquad -\log P_x e^{-Z} = \lim \ V_{D_n}(c_1 u_1 + c_2 u_2)(x).$$

It follows from 2.3.C and (2.22) that

$$(3.17) \qquad V_{D_n}(c_1 u_1 + c_2 u_2) \le (c_1 \vee 1)u_1 + (c_2 \vee 1)u_2.$$

Since $\mu \in \mathcal{M}^* \subset \mathcal{M}_0^*$, the right side in (3.17) is finite and, by the dominated convergence theorem,

$$(3.18) \qquad \int \lim \ V_{D_n}(c_1 u_1 + c_2 u_2)(x)\mu(dx) \le \lim \langle V_{D_n}(c_1 u_1 + c_2 u_2), \mu \rangle.$$

By (3.15), (3.16), (3.18) and (3.17), condition $CB(\mathcal{M}^*)$ holds for Z. By (3.16) and (3.17), Z satisfies FIN. Hence $Z \in \mathfrak{Z}(\mathcal{M}^*)$. By (3.10), u is a subadditive function. $\qquad \square$

Theorem 3.4. *Let $\mathcal{M}_n^*$ be total classes and let $\mathcal{M}^*$ be the intersection of the $\mathcal{M}_n^*$. Suppose $Z_n \in \mathfrak{Z}(\mathcal{M}_n^*)$ and $Z_n \uparrow Z$ P_μ-a.s. for all $\mu \in \mathcal{M}^*$. Then $Z \in \mathfrak{Z}(\mathcal{M}^*)$, and the log-potential u_n of Z_n converges to the log-potential u of Z.*

Proof. Clearly, $Z \in \mathcal{F}(E', \mathcal{M}^*)$. If $\mu \in \mathcal{M}^*$, then (3.6) holds for all Z_n and, by the monotone convergence theorem and the dominated convergence theorem, it holds for Z. By the dominated convergence theorem, $u_n \uparrow u$. By Theorem 3.2, $u_n \in \mathcal{U}$ and, by 2.1.C, $u \in \mathcal{U}$. Since $u < \infty$, Z satisfies FIN. $\qquad \square$

4. Solutions determined by continuous linear additive functionals. (Γ, ν)-solutions

4.1. Continuous linear additive functionals. Let (X_t, P_μ) be a superdiffusion in E and let $\mathcal{M}^*$ be a total subset of $\mathcal{M}$. Denote by $\mathcal{F}$ the σ-algebra generated by $X_s, s < \infty$, and by $\mathcal{F}_t$ the σ-algebra generated by $X_s, s \le t$. A function $A_t(\omega)$ from $[0, \infty] \times \Omega$ to $[0, \infty]$ is called a *continuous additive functional of X with determining set* $\mathcal{M}^*$ if:

4.1.A. $A_0 = 0$.

4.1.B. For every t and every μ, A_t is measurable with respect to the P_μ-completion of $\mathcal{F}$; it is also measurable with respect to P_μ-completion of $\mathcal{F}_t$ if $\mu \in \mathcal{M}^*$.

4.1.C. $A_{s+t} = A_s + \theta_s A_t$ P_μ-a.s. for all $\mu \in \mathcal{M}^*$ and all pairs s, t. (Here θ_s are the shift operators for X.)

4.1.D. A_t is continuous in t for P_μ-almost all ω for every $\mu \in \mathcal{M}^*$.

If the A^n are continuous additive functionals with determining sets $\mathcal{M}_n^*$, then $A_t = A_t^1 + \cdots + A_t^n + \ldots$ is a continuous additive functional with determining set $\bigcap \mathcal{M}_n^*$.

Let A and $\tilde{A}$ be continuous additive functionals with determining sets $\mathcal{M}^*$ and $\tilde{\mathcal{M}}^*$. We say that functionals A and $\tilde{A}$ are *equivalent* and we write $A \sim \tilde{A}$ if $P_\mu\{A_t = \tilde{A}_t\} = 1$ for all t and all $\mu \in \mathcal{M}^* \cap \tilde{\mathcal{M}}^*$. We say that A and $\tilde{A}$ are *indistinguishable* if, in addition, $\mathcal{M}^* = \tilde{\mathcal{M}}^*$.

We call

$$(4.1) \qquad h(x) = P_x A_\infty$$

the *potential* and

$$(4.2) \qquad u(x) = -\log P_x e^{-A_\infty}$$

the *log-potential* of A. [In the terminology of Section 3.1, u is the log-potential of A_∞.]

We say that a continuous additive functional A is *linear* if

$$(4.3) \qquad\qquad P_\mu A_\infty = \langle h, \mu \rangle$$

and

$$(4.4) \qquad\qquad P_\mu e^{-A_\infty} = e^{-\langle u, \mu \rangle}$$

for all μ in determining set $\mathcal{M}^*$. [Formula (4.4) means that A_∞ satisfies CB($\mathcal{M}^*$).]

Suppose that the potential h of a continuous linear additive (CLA) functional A is an L-harmonic function h. It follows from Theorem 1.2 in [10] that the log-potential u of A satisfies equation

$$(4.5) \qquad\qquad u + \mathcal{E}(u) = h \quad \text{in } E,$$

where $\mathcal{E}(u) = \mathcal{E}_E(u) = G(u^\alpha)$ (cf. (1.10)). Moreover, by Theorem 2.1 in [9], u is the unique solution of (4.5). By 2.1.A, $u \in \mathcal{U}$.

4.2. Classes H^* and H^{**}. Denote by H^* the class of all positive L-harmonic functions h such that equation (4.5) has a positive solution u. Put $h \in H^{**}$ if $h \in H^*$ and if $\mathcal{E}(h)(x) < \infty$ for some x (which implies that $\mathcal{E}(h)(x) < \infty$ for all x). It follows from [8, Theorem 3.1 and proof of Theorem 2.2] (cf. [10, 4.1.B]) that every $h \in H^*$ can be represented in the form

$$(4.6) \qquad\qquad h = h_1 + \cdots + h_n + \ldots \quad \text{with } h_n \in H^{**}.$$

Theorems 1.7, 1.1, 1.3 in [11] imply

Theorem 4.1. *If $h \in H^{**}$, then there exists a unique (up to indistinguishability) CLA functional A of X with potential h and determining set $\mathcal{M}_\mathcal{E}(h)$. The log-potential u of A is a unique solution of equation (4.5).*

Remark. If A^n are CLA functionals corresponding to $h_n \in H^{**}$ by Theorem 4.1 and if $h = h_1 + \cdots + h_n + \cdots \in H^{**}$, then $A^1 + \cdots + A^n + \ldots$ is equivalent to the CLA functional corresponding to h.

4.3. Continuous linear additive functional with spectral measure ν. Denote by $\mathcal{N}^*$ the set of all measures ν on E' such that $K\nu \in H^*$. Analogously, $\nu \in \mathcal{N}^{**}$ if $K\nu \in H^{**}$. Put $\nu \in \mathcal{N}$ if there exist $\nu_n \in \mathcal{N}^{**}$ such that

$$(4.7) \qquad\qquad \nu = \nu_1 + \cdots + \nu_n + \ldots.$$

By (4.6), $\mathcal{N} \supset \mathcal{N}^*$.

Remark 1. Clearly, every $\nu \in \mathcal{N}$ is Σ-finite.

Remark 2. If $\nu^n \in \mathcal{N}^{**}$ and $\nu^n \uparrow \nu \in \mathcal{N}^{**}$, then, by the Remark to Theorem 4.1, $\lim A^{\nu_n} \sim A^\nu$.

Theorem 4.2. *It is possible to define for every $\nu \in \mathcal{N}$ a CLA functional A^ν of X with determining set $\mathcal{M}^*(\nu)$ in such a way that:*

4.3.A. If $\nu = \nu_1 + \cdots + \nu_n + \ldots$, then A^ν is equivalent to $A^{\nu_1} + \cdots + A^{\nu_n} + \ldots$.

4.3.B. If $\nu \in \mathcal{N}^{**}$ and if $h = K\nu$, then A^ν is indistinguishable from the CLA functional corresponding to $h = K\nu$ by Theorem 4.1.

Proof. 1°. We use Theorem 4.1 to define A^ν for $\nu \in \mathcal{N}^{**}$, and we define A^ν as the sum of the A^{ν_n} if ν is defined by (4.7). To justify this definition, we prove that, if $\nu = \sum \nu_i = \sum \tilde{\nu}_j$, then $\sum A^{\nu_i} \sim \sum A^{\tilde{\nu}_j}$. Indeed, since the measure ν is Σ-finite, there exists a finite measure μ such that $\nu(dx) = \rho(x)\mu(dx)$ for a positive function ρ. Put

$$\nu^m = \nu_1 + \cdots + \nu_m, \qquad \tilde{\nu}^n = \tilde{\nu}_1 + \cdots + \tilde{\nu}_n.$$

By the Radon-Nikodým theorem, $\nu^m(dx) = \rho^m(x)\mu(dx)$, $\tilde{\nu}^n(dx) = \tilde{\rho}^n(x)\mu(dx)$ with $\lim \rho^m = \lim \tilde{\rho}^n = \rho$ μ-a.e. The measures $\nu^{mn}(dx) = [\rho^m(x) \wedge \tilde{\rho}^n(x)]\mu(dx)$ belong to $\mathcal{N}^{**}$. The corresponding functionals $A^{mn} = A^{\nu^{mn}}$ are monotone increasing in m, n, and therefore

(4.8) $$\lim_m \lim_n A^{mn} = \sup_{m,n} A^{mn} = \lim_n \lim_m A^{mn}.$$

Since $\nu^{mn} \uparrow \nu^m$ as $n \to \infty$, we have $\lim_n A^{mn} \sim A^{\nu^m}$ by Remark 2. Clearly, $A^{\nu^m} \sim A^{\nu_1} + \cdots + A^{\nu_m}$, and therefore the left side in (4.8) is equal to $\sum A^{\nu_i}$. Analogously, the right side is equal to $\sum \tilde{A}^{\nu_j}$.

2°. Properties 4.3.A and B hold for $\nu \in \mathcal{N}^{**}$ by Theorem 4.1, and they can be obtained for $\nu \in \mathcal{N}$ by a passage to the limit. $\qquad\square$

We call A^ν the *CLA functional with spectral measure* ν. If ν has the form (4.7), then $\bigcap \mathcal{M}_\mathcal{E}^*(\nu_n)$ is a determining set of A^ν.

A necessary condition for a measure ν to belong to $\mathcal{N}$ is given by

Theorem 4.3. *No measures $\nu \in \mathcal{N}$ charge $\mathcal{R}$-polar sets.*

Proof. For finite measures ν, this follows from [8, Theorem 3.1]. An extension to arbitrary $\nu \in \mathcal{N}$ is obvious because all $\nu \in \mathcal{N}$ are Σ-finite. $\qquad\square$

Theorem 4.4. *If A^ν is a CLA functional with the spectral measure ν and if $\mathcal{M}^*$ is the determining set of A^ν, then A_∞^ν is an element of class $\mathfrak{Z}(\mathcal{M}^*)$, and the log-potential of A^ν belongs to $\mathcal{U}$.*

Proof. By Theorem 3.4, it is sufficient to consider $\nu \in \mathcal{N}^{**}$. Let h and u be the potential and the log-potential of A^ν, and let $\mu \in \mathcal{M}^*$. By Lemma 5.2 in [10], A_∞^ν is measurable with respect to the P_μ-completion of $\mathcal{F}_{\supset D}$ for every $D \in \mathcal{O}$ and it is measurable with respect to the P_μ-completion of $\bigcap \mathcal{F}_{\supset D_n}$ for every standard sequence D_n approximating E. By Theorem 1.2 in [10], $u \in \mathcal{U}$. By (1.12) and (4.4), P_μ-a.s.,

$$P_\mu\{e^{-A_\infty^\nu} | \mathcal{F}_{\subset D_n}\} = P_{X_{D_n}} e^{-A_\infty^\nu} = e^{-\langle u, X_{D_n}\rangle},$$

and therefore

$$e^{-A_\infty^\nu} = e^{-\lim\langle u, X_{D_n}\rangle}.$$

By comparing this formula with (3.2), we note that A_∞^ν is $\mathcal{M}^*$-equivalent to a stochastic boundary value of u. It belongs to $\mathfrak{Z}(\mathcal{M}^*)$ by Theorem 3.2. $\qquad\square$

4.5. (Γ, ν)-solutions. For every closed set $\Gamma \subset E'$ and every $\nu \in \mathcal{N}$ we set

(4.9) $$u(x) = -\log P_x\{\mathcal{R} \cap \Gamma = \emptyset, e^{-A_\infty^\nu}\}.$$

We prove that $u \in \mathcal{U}$, and we call u the (Γ, ν)-*solution*.

We need

Lemma 4.1. *Let*

$$(4.10) \qquad\qquad U_n = \{x : d(x, \Gamma) > 1/n\}.$$

Then, for every $\mu \in \mathcal{M}$,

$$(4.11) \qquad\qquad \{X_{U_n} = 0\} \uparrow \{\mathcal{R} \cap \Gamma = \emptyset\} \quad P_\mu\text{-}a.s.$$

Proof. It follows from (1.12) that, P_μ-a.s.,

$$\{X_{U_n} = 0\} \subset \{X_{U_{n+1}} = 0\}.$$

By [3, Lemma 2.1],

$$\{\mathcal{R} \subset \bar{U}_{n-1}\} \subset \{X_{U_n} = 0\} \subset \{\mathcal{R} \subset \bar{U}_n\} \quad P_\mu\text{-a.s.}$$

Clearly, this implies (4.11). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

Theorem 4.5. *If Γ is a closed subset of E', then*

$$(4.12) \qquad\qquad Z(\Gamma) = \begin{cases} 0 & if \quad \mathcal{R} \cap \Gamma = \emptyset, \\ \infty & if \quad \mathcal{R} \cap \Gamma \neq \emptyset \end{cases}$$

belongs to class $\mathfrak{Z}(\mathcal{M}_0^*)$.

If A^ν is a CLA functional with spectral measure ν and determining set $\mathcal{M}^*$, then

$$(4.13) \qquad\qquad Z = Z(\Gamma) + A_\infty^\nu$$

belongs to $\mathfrak{Z}(\mathcal{M}_1^*)$, where $\mathcal{M}_1^* = \mathcal{M}_0^* \cap \mathcal{M}^*$.

Proof. For every $D \in \mathcal{O}$, there exists m such that $U_n \supset D$ for all $m \geq n$. By (4.11), $Z(\Gamma) \in \mathcal{F}(E', \mathcal{M})$. Condition FIN holds for $Z(\Gamma)$, because $P_x\{\mathcal{R} \cap E' = \emptyset\} > 0$ for every x by [4, Theorem 2.1]. Condition CB($\mathcal{M}$) holds for $Z_n = 0$ if $X_{U_n} = 0$ and $Z_n = \infty$ otherwise by [5, II.4.6]. Condition CB($\mathcal{M}_0^*$) for $Z(\Gamma)$ follows from (4.11) and the dominated convergence theorem.

The second statement of Theorem 4.5 is an implication of the first statement and Theorem 3.3. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

5. TRACE OF SOLUTIONS

5.1. Moderate solutions. We say that $u \in \mathcal{U}$ is a *moderate solution* if it is dominated by an L-harmonic function. It follows from Theorem 1.3 in [8] and 2.1.A that:

5.1.A. If u is a moderate solution, then $h = u + \mathcal{E}(u)$ is the minimal L-harmonic majorant of u and u is the maximal solution dominated by h.

The *trace of a moderate solution* u is defined as the trace of the L-harmonic function $h = u + \mathcal{E}(u)$. By [8, Theorem 3.1],

5.1.B. If ν is the trace of a moderate solution u, then $\nu(B) = 0$ for all $\mathcal{R}$-polar sets B.

5.1.C. The log-potential u of $Z \in \mathfrak{Z}(\mathcal{M}^*)$ is a moderate solution if and only if

$$(5.1) \qquad\qquad h(x) = P_x Z < \infty \qquad \text{for all} \ \ x \in E.$$

If u is moderate, then its minimal L-harmonic majorant is given by (5.1).

Proof. For every $D \in \mathcal{O}$, by (1.12) and (1.13),

$$h(x) = P_x P_{X_D} Z = P_x \langle h, X_D \rangle = \Pi_x h(\xi_\tau),$$

where τ is the first exit time from D. Therefore, if $h < \infty$, then h is L-harmonic. By Jensen's inequality, $P_x e^{-Z} \geq e^{-h(x)}$ and therefore $u \leq h$. Hence (5.1) implies that u is moderate.

Let D_n be a standard sequence approximating E and let τ_n be the first exit time from D_n. It follows from (3.2) and Fatou's lemma that

$$h(x) \leq \lim P_x \langle u, X_{D_n} \rangle.$$

If u is moderate and if it is dominated by an L-harmonic function $\tilde{h}$, then

$$P_x \langle u, X_{D_n} \rangle \leq P_x \langle \tilde{h}, X_{D_n} \rangle = \Pi_x \tilde{h}(\xi_{\tau_n}) \leq \tilde{h}(x).$$

Hence $h < \infty$ and it is the minimal L-harmonic majorant of u. $\qquad \square$

Recall that, according to Section 4.2, H^* is the set of all L-harmonic functions h of the form $h = u + \mathcal{E}(u)$ with positive u.

5.1.D. The class H^* is a convex cone, and it contains with every h all positive L-harmonic functions dominated by h.

Proof. By Lemma 1.2 in [8], to every L-harmonic function h there corresponds a function Z_h such that

$$Z_h = \lim \langle h, X_{D_n} \rangle \quad \text{a.s.}$$

for every standard sequence D_n approximating E. Moreover,

$$(5.2) \qquad\qquad\qquad P_x Z_h \leq h(x).$$

By Theorem 1.5 in [8], (5.2) holds with the equality sign for $h \in H^*$. Lemma 5.3 in [10] in combination with Theorem 4.2 in [8] yields that, if (5.2) holds with the equality sign for an L-harmonic h, then $h \in H^*$. If this condition holds for h_1, h_2, then it holds for $c_1 h_1 + c_2 h_2$ with positive constants c_1, c_2.

If h_1, h_2 are L-harmonic and if $h = h_1 + h_2 \in H^*$, then $P_x Z_h = h(x)$, $P_x Z_{h_1} \leq h_1(x)$ and $P_x Z_{h_2} \leq h_2(x)$. Since $Z_h = Z_{h_1} + Z_{h_2}$, we get $P_x Z_{h_1} = h_1(x)$ and therefore $h_1 \in H^*$. $\qquad\square$

Remark. It follows from 5.1.D and Minkowski's inequality that the class H^{**} is a cone with the same properties as H^*.

Theorem 5.1. *Let $\nu \in \mathcal{N}$. The following three conditions are equivalent:*
  (i) *ν is finite;*
  (ii) *the potential h of A^ν is finite;*
  (iii) *the log-potential u of A^ν is a moderate solution.*

Proof. The equivalence of (i) and (ii) is an implication of the relation

$$h(x) = P_x A_\infty^\nu = \int k(x, y) \nu(dy),$$

which follows from 4.3.B if $\nu \in \mathcal{N}^{**}$ and which can be obtained by a monotone passage to the limit for an arbitrary $\nu \in \mathcal{N}$. The equivalence of (ii) and (iii) follows from 5.1.C. $\qquad\square$

For every closed subset B of the Martin boundary E', we denote by $\mathcal{O}(B)$ the class of all open subsets D of E such that the M-closure of D is disjoint from B.

Theorem 5.2 (Extended mean value property). *Let u be a moderate solution with trace ν. If ν is concentrated on a closed subset B of E', then*

$$(5.3) \qquad u(x) = -\log P_x e^{-\langle u, X_D \rangle} \quad \text{in } D$$

for every $D \in \mathcal{O}(B)$.

The proof is based on

Lemma 5.1. *Suppose $\tilde{D}_m = D_m \cap D$, where D_m is a standard sequence approximating E and $D \in \mathcal{O}(B)$. If $u \in \mathcal{U}$ is dominated by*

$$(5.4) \qquad h_B(x) = \int_B k(x, y)\nu(dy),$$

then, for every $\mu \in \mathcal{M}$,

$$(5.5) \qquad P_\mu \langle u 1_D, X_{\tilde{D}_m} \rangle \to 0.$$

Proof. Let τ and τ_m be the first exit times from D and from $\tilde{D}_m$. By (1.13) and (1.5),

$$P_\mu \langle u 1_D, X_{\tilde{D}_m} \rangle = \Pi_\mu (u 1_D)(\xi_{\tau_m}) \leq \Pi_\mu (h_B 1_D)(\xi_{\tau_m}) = \Pi_\mu^{h_B} \{ \xi_{\tau_m} \in D \}.$$

Denote by C the intersection of E' with the M-closure of D. Note that C is disjoint from B and $\{\xi_{\tau_m} \in D\} \downarrow \{\tau = \zeta, \xi_{\zeta-} \in C\}$. By (1.4),

$$\Pi_\mu^{h_B} \{ \xi_{\tau_m} \in D \} \to \Pi_\mu^{h_B} \{ \tau = \zeta, \xi_{\zeta-} \in C \} \leq \int_{C \cap B} k(x, y)\nu(dy) = 0.$$

$$\square$$

Proof of Theorem 5.2. Let D_m and $\tilde{D}_m$ be the sets introduced in Lemma 5.1. By the mean value property 2.1.D,

$$(5.6) \qquad u(x) = -\log P_x e^{-\langle u, X_{\tilde{D}_m} \rangle} \quad \text{in } \tilde{D}_m.$$

Note that $\partial \tilde{D}_m = B_m \cup C_m$, where $B_m = \partial \tilde{D}_m \cap \partial D$, and $C_m = \partial \tilde{D}_m \cap D$ and therefore, for every $x \in \tilde{D}_m$,

$$\langle u, X_{\tilde{D}_m} \rangle = \langle u 1_{B_m}, X_{\tilde{D}_m} \rangle + \langle u 1_{C_m}, X_{\tilde{D}_m} \rangle = \langle u, X_m \rangle + \langle u 1_D, X_{\tilde{D}_m} \rangle \quad P_x\text{-a.s.},$$

where X_m is the restriction of $X_{\tilde{D}_m}$ to ∂D. By Lemma 5.1, the second term tends to 0 in P_μ-probability as $m \to \infty$. Formula (5.3) will follow from (5.6) if we prove that, for every $x \in D$,

$$(5.7) \qquad \lim \langle u, X_m \rangle = \langle u, X_D \rangle \quad P_x\text{-a.s.}$$

By Lemma 3.1 of [9], $X_m \leq X_{m+1} \leq X_D$ P_x-a.s., and therefore

$$(5.8) \qquad X_m \uparrow X^* \leq X_D \quad P_x\text{-a.s.}$$

By (1.13),

$$P_x \langle 1, X_m \rangle = \Pi_x \{ \xi_{\tau_m} \in \partial D \},$$
$$P_x \langle 1, X_D \rangle = \Pi_x \{ \xi_\tau \in \partial D \},$$

where τ_m, τ are the first exit moments from $\tilde{D}_m, D$. Since $\{\xi_{\tau_m} \in \partial D\} \uparrow \{\xi_\tau \in \partial D\}$, (5.8) implies that $X^* = X_D$ P_x-a.s. $\qquad \square$

5.2. Operators Q_B. To extend the definition of the trace to all $u \in \mathcal{U}$, we construct a family of operators $Q_B : \mathcal{U} \to \mathcal{U}$.

Let $\mathcal{P}$ be a partially ordered set. We say that a sequence $p_n \in \mathcal{P}$ *supports* $\mathcal{P}$ if $p_1 \prec p_2 \prec \cdots \prec p_n \prec \ldots$ and if, for every $p \in \mathcal{P}$, there exists N such that $p \prec p_n$ for all $n \geq N$. If φ is a positive monotone increasing function on $\mathcal{P}$, then, for every sequence $\{p_n\}$ supporting $\mathcal{P}$, the limit $\lim \varphi(p_n)$ exists and does not depend on the choice of $\{p_n\}$. We call it the *limit of φ along $\mathcal{P}$*.

The set $\mathcal{O}(B)$ introduced in Section 5.1 is a partially ordered set relative to the inclusion. By Theorem 2.2, for every $u \in \mathcal{U}$ and every $x \in E$, $V_D(u)(x)$ is a monotone decreasing function on $\mathcal{O}(B)$, and therefore there exists a pointwise limit of $V_D(u)$ along $\mathcal{O}(B)$. We denote it $Q_B(u)$.

For every closed set $B \subset E'$ and for every $\varepsilon > 0$ we put

$$(5.9) \qquad D(B, \varepsilon) = \{x \in E : d(x, B) > \varepsilon\}.$$

If $\varepsilon_n \downarrow 0$, then $D(B, \varepsilon_n)$ supports $\mathcal{O}(B)$. Clearly,

$$(5.10) \qquad Q_B(u) = \lim_{\varepsilon \to 0} V_{D(B,\varepsilon)}(u).$$

Note that a sequence D_n supports $\mathcal{O}(E')$ if and only if it is a standard sequence approximating E.

Property 2.1.C implies that $Q_B(u) \in \mathcal{U}$. Operators Q_B have the following properties:

5.2.A. $Q_B(u_1) \leq Q_B(u_2)$ for $u_1 \leq u_2$.

5.2.B. $Q_B(u) \leq u$.

5.2.C. If $B_1 \supset B_2$, then $Q_{B_1}(u) \geq Q_{B_2}(u)$.

5.2.D. For every B_1, B_2, $\quad Q_{B_1 \cup B_2}(u) \leq Q_{B_1}(u) + Q_{B_2}(u)$.

5.2.E. $Q_{E'}(u) = u$.

5.2.F. If $u \leq u_1 + u_2$, then $Q_B(u) \leq Q_B(u_1) + Q_B(u_2)$.

Property 5.2.A follows from 2.3.A; 5.2.B follows from (2.17); 5.2.C follows from (2.18); 5.2.D is an implication of 2.3.D and the relation $D(B_1 \cup B_2, \varepsilon) = D(B_1, \varepsilon) \cap D(B_2, \varepsilon)$. 5.2.E holds because, by 2.1.D, $V_{D(E',\varepsilon)}(u) = u$ for all ε. Finally, 5.2.F follows from 2.3.C and 2.3.A.

5.2.G. For every moderate solution u, the trace of $u_B = Q_B(u)$ is concentrated on B.

Proof. If ν is the trace of u, then u is dominated by h defined by (1.1). Let h_B be given by (5.4). By (1.4) and (1.5),

$$(5.11) \qquad h_B(x) = \Pi_x^h\{\xi_{\zeta-} \in B\} = \lim_{\varepsilon \to 0} \Pi_x^h\{\tau_\varepsilon < \zeta\} = \lim_{\varepsilon \to 0} \Pi_x h(\xi_{\tau_\varepsilon}),$$

where τ_ε is the first exit time from $D(B, \varepsilon)$. By (2.16),

$$(5.12) \qquad V_{D(B,\varepsilon)}(u)(x) \leq \Pi_x u(\xi_{\tau_\varepsilon}) \leq \Pi_x h(\xi_{\tau_\varepsilon}).$$

By (5.10), (5.11) and (5.12), $u_B \leq h_B$. Hence the trace of u_B is concentrated on B. $\qquad\square$

5.2.H. If u is a moderate solution with the trace concentrated on B, then

$$(5.13) \qquad Q_B(u) = u.$$

This follows from Theorem 5.2 and (5.10).

5.2.I. If $Q_B(u)$ is moderate and if $B \subset B_0$, then $Q_{B_0}[Q_B(u)] = Q_B(u) = Q_B[Q_{B_0}(u)]$.

Indeed, by 5.2.G, the trace of $Q_B(u)$ is concentrated on B and the first part of 5.2.I follows from 5.2.H. In particular, $Q_B[Q_B(u)] = Q_B(u)$. On the other hand, by 5.2.C and 5.2.A, $Q_B[Q_{B_0}(u)] \geq Q_B[Q_B(u)] = Q_B(u)$ and $Q_B[Q_{B_0}(u)] \leq Q_B(u)$ by 5.2.B and 5.2.A.

5.3. Stochastic boundary value on B. The following result can be proved by the same arguments as Theorem 3.1:

Theorem 5.3. *For every $u \in \mathcal{U}$ and every closed subset B of E', there exists a function Z_B such that*

$$(5.14) \qquad \lim \langle u, X_{D_n} \rangle = Z_B \quad P_\mu\text{-a.s.}$$

for every sequence D_n supporting $\mathcal{O}(B)$ and every $\mu \in \mathcal{M}$. For all $\mu \in \mathcal{M}$,

$$(5.15) \qquad P_\mu e^{-Z_B} = e^{-\langle Q_B(u), \mu \rangle}.$$

In particular,

$$(5.16) \qquad Q_B(u)(x) = -\log P_x e^{-Z_B}.$$

Corollary. *For every $\mu \in \mathcal{M}$ and every B,*

$$(5.17) \qquad \{\mathcal{R} \cap B = \emptyset\} \subset \{Z_B = 0\} \quad P_\mu\text{-a.s.}$$

If B is $\mathcal{R}$-polar, then $Q_B(u) = 0$.

Indeed, (5.17) follows from (5.14) and Lemma 4.1. If B is $\mathcal{R}$-polar, then $Z_B = 0$ P_μ-a.s., and $Q_B(u) = 0$ by (5.16).

We call Z_B a stochastic boundary value of the solution u on set B.

5.4. Expression of Z_B in terms of Q_B.

Theorem 5.4. *Suppose that $u_B = Q_B(u)$ is a moderate solution. Then, for every standard sequence D_n approximating E and every $\mu \in \mathcal{M}$,*

$$(5.18) \qquad Z_B = \lim \langle u_B, X_{D_n} \rangle \quad P_\mu\text{-a.s.}$$

Proof. 1°. By Theorem 5.3, there exist functions $Z_B, \tilde{Z}_B$ and $\hat{Z}_B$ such that

$$(5.19) \qquad Z_B = \lim \langle u, X_{\tilde{D}_n} \rangle \quad P_\mu\text{-a.s.},$$

$$(5.20) \qquad \tilde{Z}_B = \lim \langle u_B, X_{\tilde{D}_n} \rangle \quad P_\mu\text{-a.s.}$$

for every $\tilde{D}_n$ supporting $\mathcal{O}(B)$ and every μ, and

$$(5.21) \qquad \hat{Z}_B = \lim \langle u_B, X_{D_n} \rangle \quad P_\mu\text{-a.s.}$$

for every standard sequence D_n approximating E and every μ. Moreover,

$$(5.22) \qquad P_\mu e^{-Z_B} = P_\mu e^{-\tilde{Z}_B} = P_\mu e^{-\hat{Z}_B} = e^{-\langle u_B, \mu \rangle}$$

since $Q_B(u_B) = u_B$ by 5.2.I.

Formula (5.18) will follow from (5.22) if we show that $\hat{Z}_B \leq \tilde{Z}_B \leq Z_B$ P_μ-a.s. The second inequality follows from 5.2.B. It remains to prove that

$$(5.23) \qquad \hat{Z}_B \leq \tilde{Z}_B \quad P_\mu\text{-a.s.}$$

$2°$. Consider $D_{mn} = D_m \cap \tilde{D}_n$, where D_m supports $\mathcal{O}(E')$ and $\tilde{D}_n$ supports $\mathcal{O}(B)$. By Lemma 5.1, $P_\mu \langle u_B 1_{\tilde{D}_n}, X_{D_{mn}} \rangle \to 0$ as $m \to \infty$, and therefore there exists a monotone increasing sequence $m_n \to \infty$ such that

$$(5.24) \qquad P_\mu \langle u_B 1_{\tilde{D}_n}, X_{D_{m_n n}} \rangle < 1/n.$$

The sequence $D_n^* = D_{m_n n}$ supports $\mathcal{O}(E')$.

Note that $\partial D_n^* = B_n \cup C_n$, where $B_n = \partial D_n^* \cap \partial \tilde{D}_n$ and $C_n = \partial D_n^* \cap \tilde{D}_n$. By [9, Lemma 3.1], $X_{D_n^*} \leq X_{\tilde{D}_n}$ on B_n, and therefore

$$\langle u_B, X_{D_n^*} \rangle \leq \langle u_B 1_{B_n}, X_{\tilde{D}_n} \rangle + \langle u_B 1_{C_n}, X_{D_n^*} \rangle \leq \langle u_B, X_{\tilde{D}_n} \rangle + \langle u_B 1_{\tilde{D}_n}, X_{D_n^*} \rangle.$$

By (5.21), (5.20) and (5.24), this implies (5.18). $\qquad \square$

It follows from Theorem 5.3 and 5.2.C, D that:

5.4.A. If $B_1 \supset B_2$, then $Z_{B_1} \geq Z_{B_2}$ P_μ-a.s. for all $\mu \in \mathcal{M}$.

5.4.B. For every B_1, B_2, $Z_{B_1 \cup B_2} \leq Z_{B_1} + Z_{B_2}$ P_μ-a.s. for all $\mu \in \mathcal{M}$.

5.5. Trace of $Q_B(u)$.

Lemma 5.2. *If u is a moderate solution, then, for every B, $u_B = Q_B(u)$ is also moderate and the trace of u_B is the restriction of the trace of u to B. If the trace of u does not charge B, then $u_B = 0$.*

Proof. Consider the minimal L-harmonic majorant

$$h(x) = \int_{E^*} k(x,y) \nu(dy)$$

of u and the minimal L-harmonic majorant h_B^* of u_B. By 5.2.G, u_B is dominated by h_B given by (5.4), and therefore

$$(5.25) \qquad h_B^* \leq h_B.$$

By 5.2.B, there exists $\tilde{u}$ such that $\tilde{u} + \mathcal{E}(\tilde{u}) = h_B$. The trace of $\tilde{u}$ is concentrated on B and, by (5.13), $\tilde{u}_B = Q_B(\tilde{u}) = \tilde{u}$. Hence $\tilde{h}_B = \tilde{u}_B + \mathcal{E}(\tilde{u}_B) = h_B$. By 5.1.A, $\tilde{u} \leq u$ because $h_B \leq h$. By 5.2.A, $\tilde{u}_B \leq u_B$ and therefore

$$(5.26) \qquad h_B = \tilde{h}_B \leq h_B^*.$$

By (5.25) and (5.26), $h_B^* = h_B$. $\qquad \square$

5.6. Properties of w_Γ.
For every closed subset Γ of E' we put

$$(5.27) \qquad w_\Gamma(x) = -\log P_x \{\mathcal{R} \cap \Gamma = \emptyset\}.$$

5.6.A. We have

$$(5.28) \qquad Q_\Gamma(w_\Gamma) = w_\Gamma.$$

Proof. By Theorem 8.2 in [5], for every measure μ with support disjoint from Γ,

$$(5.29) \qquad P_\mu \{\mathcal{R} \cap \Gamma = \emptyset\} = e^{-\langle w_\Gamma, \mu \rangle}.$$

Let D_n be a supporting sequence of $\mathcal{O}(\Gamma)$. By the definition of Q_Γ, (2.15) and (5.29),

$$Q_\Gamma(w_\Gamma) = -\lim \log P_x P_{X_{D_n}} \{\mathcal{R} \cap \Gamma = \emptyset\}.$$

By (4.11), $\{\mathcal{R} \cap \Gamma = \emptyset\}$ belongs to the P_x-completion of $\mathcal{F}_{\supset D_n}$; and, by (1.12),

$$P_x P_{X_{D_n}} \{\mathcal{R} \cap \Gamma = \emptyset\} = P_x \{\mathcal{R} \cap \Gamma = \emptyset\} = e^{-w_\Gamma(x)}. \qquad \square$$

5.6.B. If w_Γ is a moderate solution, then Γ is $\mathcal{R}$-polar.

Indeed, w_Γ is the log-potential of $Z(\Gamma)$ given by (4.12). By 5.1.C, $h(x) = P_x Z(\Gamma) < \infty$, which implies 5.6.B.

5.6.C. For every $u \in \mathcal{U}$ and every closed $B \subset \Gamma$,

$$(5.30) \qquad\qquad Q_B(u) \leq w_\Gamma.$$

Indeed, by (5.16) and (5.17),

$$(5.31) \quad Q_\Gamma(u)(x) = -\log P_x e^{-Z_\Gamma} \leq -\log P_x\{Z_\Gamma = 0\} \leq -\log P_x\{\mathcal{R} \cap \Gamma = \emptyset\}.$$

(5.30) follows from (5.31) by 5.2.C.

5.6.D. If the spectral measure ν of a CLA A is concentrated on Γ and if u is the log-potential of A, then

$$(5.32) \qquad\qquad u = Q_\Gamma(u) \leq w_\Gamma.$$

Proof. Note that, if $u_n = Q_\Gamma(u_n)$ and if $u_n \uparrow u$, then $Q_\Gamma(u) = u$ by 5.2.A and 5.2.B. Therefore it is sufficient to prove (5.32) for $\nu \in \mathcal{N}^{**}$. By Theorem 5.1, in this case u is moderate. By Lemma 5.2, $u_\Gamma = Q_\Gamma(u)$ is a moderate solution with the same trace as u, and therefore $u_\Gamma = u$. The second part of (5.32) follows from 5.6.A. $\qquad\square$

5.6.E. If ν is a finite measure concentrated on Γ and if u is the log-potential of A^ν, then A^ν_∞ is $\mathcal{M}^*$-equivalent to the stochastic boundary value Z_Γ of u on Γ.

Indeed, let $\mu \in \mathcal{M}^*$ and let D_n be a standard sequence approximating E. By Theorems 4.4 and 3.2,

$$(5.33) \qquad\qquad A^\nu_\infty = \lim\langle u, X_{D_n}\rangle \quad P_\mu\text{-a.s.}$$

By Theorem 5.3,

$$(5.34) \qquad\qquad Z_\Gamma = \lim\,\langle u_\Gamma, X_{D_n}\rangle \quad P_\mu\text{-a.s.},$$

and $P_\mu\{A^\nu_\infty = Z_\Gamma\} = 1$ by 5.6.D.

5.7. General definition of the trace. Fix a solution u. We say that a compact set $B \subset E'$ is *moderate for* u if the solution $u_B = Q_B(u)$ is moderate. Let ν_B stand for the trace of u_B. By 5.2.D, the union of two moderate sets is moderate. Suppose that B is moderate, and let $\tilde{B} \subset B$. By Lemma 5.2, $\tilde{B}$ is moderate. By 5.2.I, $Q_B Q_{\tilde{B}}(u) = Q_{\tilde{B}}(u)$, and, by Lemma 5.2, $\nu_{\tilde{B}}$ is the restriction of ν_B to $\tilde{B}$.

A relatively open subset A of E' is called *moderate* if all compact subsets of A are moderate. The union O of all moderate open sets is moderate. Clearly, there exists a unique measure ν on O such that its restriction to an arbitrary compact subset B coincides with ν_B. This measure is a Radon measure on O (that is, it is finite on all compact subsets). By 5.1.B, $\nu(B) = 0$ for all $\mathcal{R}$-polar sets B. We call closed set $\Gamma = E' \setminus O$ the *singular set of the solution* u, and we call the pair (Γ, ν) the *trace of* u *on* E'.

6. Trace of (Γ, ν)-solutions

6.1. Normal pairs. According to the definition in Section 5.7, the trace of every $u \in \mathcal{U}$ is a pair (Γ, ν) with the following properties:

6.1.A. Γ is a closed subset of E'.

6.1.B. ν is a Radon measure on $\mathcal{O} = E' \setminus \Gamma$.

6.1.C. ν does not charge $\mathcal{R}$-polar sets.

We say that x is an *explosion point of a measure* ν if $\nu(U) = \infty$ for every neighborhood U of x. If no explosion point belongs to a compact set B, then $\nu(B) < \infty$. Condition 6.1.B is equivalent to:

6.1.B'. $\mathcal{O}$ contains no explosion points of ν.

For every (Γ, ν) subject to conditions 6.1.A,B,C, we denote by $\mathcal{L}(\Gamma, \nu)$ the class of all $\mathcal{R}$-polar relatively open subsets of Γ which contain no explosion points of ν. Note that if $\Lambda = \Gamma \setminus \Gamma_0 \in \mathcal{L}(\Gamma, \nu)$, then $\{\mathcal{R} \cap \Gamma = \emptyset\} = \{\mathcal{R} \cap \Gamma_0 = \emptyset\}$ P_x-a.s. for all $x \in E$, and therefore a (Γ, ν)-solution is at the same time a (Γ_0, ν)-solution. We say that (Γ, ν) is a *normal pair* if the empty set is the only element of $\mathcal{L}(\Gamma, \nu)$.

Theorem 6.1. *Every (Γ, ν)-solution u is also a (Γ_0, ν)-solution with a normal pair (Γ_0, ν).*

Proof. The union of an arbitrary family of $\mathcal{R}$-polar sets relatively open in Γ is $\mathcal{R}$-polar. Therefore the union Λ_0 of all $\Lambda \in \mathcal{L}(\Gamma, \nu)$ belongs to $\mathcal{L}(\Gamma, \nu)$ and u is a (Γ_0, ν)-solution. Put $\Gamma_0 = \Gamma \setminus \Lambda_0$. Let $\Lambda_1 \in \mathcal{L}(\Gamma_0, \nu)$. Then $\Gamma_1 = \Gamma_0 \setminus \Lambda_1$ is closed and $\Lambda^* = \Gamma \setminus \Gamma_1 = \Lambda_0 \cup \Lambda_1$ belongs to $\mathcal{L}(\Gamma, \nu)$. Hence $\Lambda^* \subset \Lambda_0$, which implies $\Lambda_1 \subset \Lambda_0$. On the other hand, $\Lambda_1 \subset \Gamma_0 = \Gamma \setminus \Lambda_0$. Hence $\Lambda_1 = \emptyset$, and the pair (Γ_0, ν) is normal. $\qquad\square$

6.2. It follows from [12, Theorem 3.1] and 5.2.A that the following condition holds if E is a bounded domain in $\mathbb{R}^d$ with the smooth boundary and if L is a second order elliptic operator in $\mathbb{R}^d$:

($\mathbb{N}$) $Q_B[Q_\Gamma(u)] = 0$ for every $u \in \mathcal{U}$ and any disjoint closed subsets B, Γ of E'.

In Section 8 we prove a more general result which implies, in particular, that condition ($\mathbb{N}$) holds, if E is a bounded domain in $\mathbb{R}^d$, L is a second order elliptic differential operator in E with bounded coefficients, and distinct points of the Martin boundary E' correspond to distinct points of the Euclidean boundary ∂E.

Condition ($\mathbb{N}$) is equivalent to

($\mathbb{N}'$) $Q_B(W_\Gamma) = 0$ for any disjoint closed sets B, Γ.

Indeed, ($\mathbb{N}$) implies ($\mathbb{N}'$) by 5.6.A, and ($\mathbb{N}'$) implies ($\mathbb{N}$) by 5.6.C and 5.2.A.

In the rest of Section 6 and in Section 7 we consider only superdiffusions which satisfy condition ($\mathbb{N}$). Here are two immediate implication of this condition:

6.2.A. If u is the log-potential of A^ν and if ν is concentrated on Γ, then

(6.1) $$Q_B(u) = 0$$

for every B disjoint from Γ.

This follows from 5.6.D and ($\mathbb{N}'$).

6.2.B. Let u be the log-potential of A^ν. If $\nu(B) = 0$ and if $\nu(U) < \infty$ for a neighborhood U of B, then $Q_B(u) = 0$.

Proof. Denote by ν_1, ν_2 the restrictions of ν to U and to $E' \setminus U$. Let u_1 and u_2 be the log-potentials of A^{ν_1} and A^{ν_2}. By Lemma 3.1, $u \leq u_1 + u_2$, and, by 5.2.F, $Q_B(u) \leq Q_B(u_1) + Q_B(u_2)$.

By Theorem 5.1, u_1 is a moderate solution. By Lemma 5.2, $Q_B(u_1) = 0$.

By 5.6.D, $u_2 \leq w_\Gamma$, where $\Gamma = E' \setminus U$. Since B and Γ are disjoint, $Q_B(u_2) \leq Q_B(w_\Gamma) = 0$ by 5.2.A and ($\mathbb{N}$). $\qquad\square$

Lemma 6.1. *Let u be the (Γ, ν)-solution and let B be a closed subset of $O = E' \setminus \Gamma$. Then the trace of $Q_B(u)$ is equal to the restriction ν' of ν to B.*

Proof. Let $A' = A_\infty^{\nu'}$ and $A'' = A_\infty^{\nu''}$, where ν'' is the restriction of ν to $B'' = O \setminus B$. Denote by u', u'' the log-potentials of A', A''. By (4.9), $u' \leq u$, and therefore $u' = Q_B(u') \leq Q_B(u)$ by 5.2.A and 5.6.D. On the other hand, u is the log potential of $Z = A' + A'' + Z(\Gamma)$, where $Z(\Gamma)$ is given by (4.12). Therefore

$$Q_B(u) \leq Q_B(u') + Q_B(u'') + Q_B(w_\Gamma)$$

by Lemma 3.1 and 5.2.F. It remains to note that $Q_B(u'') = 0$ by 6.2.B and $Q_B(w_\Gamma) = 0$ by (ℕ). This yields $Q_B(u) = u'$. □

6.3. Trace of a (Γ, ν)-solution.

Theorem 6.2. *Let (Γ, ν) be a normal pair. If u is a (Γ, ν)-solution, then the trace of u is equal to (Γ, ν).*

Proof. Denote the trace of u by (Γ_0, ν_0). By the definition of the trace (see Section 5.7), $\mathcal{O}_0 = E' \setminus \Gamma_0$ is the maximal moderate open set, and therefore $\mathcal{O} \subset \mathcal{O}_0$ by Lemma 6.1.

If B is a closed subset of $\mathcal{O}$, then $B \subset \mathcal{O}_0$, and the trace of $Q_B(u)$ is equal to the restriction of ν_0 by the definition of ν_0. Therefore $\nu = \nu_0$ on $\mathcal{O}$. Theorem 6.2 will be proved if we show that $\Lambda = \Gamma \setminus \Gamma_0$ belongs to $\mathcal{L}(\Gamma, \nu)$. Λ contains no explosion points of ν. Indeed, since ν is concentrated on $\mathcal{O}$, every explosion point of ν is also an explosion point for ν_0 and, by the definition of the trace, it belongs to Γ_0. It remains to check that Λ is $\mathcal{R}$-polar. If B is a closed subset of Λ, then $B \subset \mathcal{O}_0$ and therefore it is moderate for u. By 5.2.A and (5.28), $Q_B(u) \geq Q_B(w_B) = w_B$. Hence, B is moderate. By Theorem 5.4, it is $\mathcal{R}$-polar. □

6.4. Trace of an arbitrary solution.

Theorem 6.3. *The trace (Γ, ν) of an arbitrary solution u is a normal pair.*

First we prove two lemmas.

Lemma 6.2. *If $D_1, D_2, \tilde{D}$ are open subsets of E and if*

$$(6.2) \qquad\qquad D_1 \cap \tilde{D} = D_2 \cap \tilde{D},$$

then

$$(6.3) \qquad \{\mathcal{R} \subset \tilde{D}\} \subset \{X_{\tilde{D}} = 0\} \subset \{X_{D_1} = X_{D_2}\} \quad P_x\text{-a.s.}$$

Proof. Let U_n be a standard sequence approximating $\tilde{D}$. By [3, Lemma 2.1], $\{\mathcal{R} \subset U_n\} \subset \{X_{\tilde{D}} = 0\}$ a.s., which implies the first part of (6.3).

A set $Q = \{X_{\tilde{D}} = 0, X_{D_1} \neq X_{D_2}\}$ belongs to $\mathcal{F}_{\supset U}$, where $U = \tilde{D} \cap D_1 = \tilde{D} \cap D_2$. By (1.12), $P_x(Q) = P_x P_{X_U}(Q)$ and, by [5, II.4.6], the second part of (6.3) will be proved if we show that $P_x(Q) = 0$ for all $x \notin U$. We have $P_x\{X_{\tilde{D}} = \delta_x\} = 1$ if $x \notin \tilde{D}$, and $P_x\{X_{D_1} = X_{D_2} = \delta_x\} = 1$ if $x \in \tilde{D} \cap U^c \subset (D_1 \cup D_2)^c$. □

Lemma 6.3. *If (Γ, ν) is the trace of u and if Γ is $\mathcal{R}$-polar and ν is finite, then u is moderate.*

Remark. According to the definition of the trace in Section 5.7, if u is moderate, then Γ is empty.

Proof. Let Z be the stochastic boundary value of u and let Z_B be its stochastic boundary value on B. Put $B_n = \{x \in E' : d(x, \Gamma) \geq 1/n\}$. Note that $B_n \uparrow O = E' \setminus \Gamma$. By (3.4) and (5.16),

$$u(x) = -\log P_x \exp\{-Z\},$$
$$u_n(x) = Q_{B_n}(x) = -\log P_x \exp\{-Z_{B_n}\}.$$

The trace of u_n is the restriction of ν to B_n, and therefore

$$u_n(x) + \mathcal{E}(u_n)(x) = \int_{B_n} k(x, y)\nu(dy) < \infty.$$

Hence, for all x and n,

$$u_n(x) \leq h(x) = \int_O k(x, y)\nu(dy).$$

By 5.4.B, there exists $\tilde{Z}$ such that $Z_{B_n} \uparrow \tilde{Z}$ P_x-a.s. for all x. We have

$$-\log P_x e^{-\tilde{Z}} = \lim u_n(x) \leq h(x).$$

To prove that u is moderate, it is sufficient to show that, for every x,

(6.4) $$\tilde{Z} = Z \quad P_x\text{-a.s.}$$

We apply Lemma 6.2 to $D_1 = D(B_n, \varepsilon), D_2 = D(E', \varepsilon), \tilde{D} = D(\Gamma, \delta)$. Condition (6.2) holds if $\varepsilon < \delta$ and $d(x, \Gamma) < \delta - \varepsilon$ for all $x \notin B_n$. We conclude from (6.3) that

$$P_x\{\mathcal{R} \subset D(\Gamma, \delta), \exp[-\langle u, X_{D(B_n, \varepsilon)}\rangle]\} = P_x\{\mathcal{R} \subset D(\Gamma, \delta), \exp[-\langle u, X_{D(E', \varepsilon)}\rangle]\}.$$

By passing to the limit, first as $\varepsilon \to 0$, then as $n \to \infty$, and finally as $\delta \to 0$, we get

$$P_x\{\mathcal{R} \cap \Gamma = \emptyset, e^{-\tilde{Z}}\} = P_x\{\mathcal{R} \cap \Gamma = \emptyset, e^{-Z}\}.$$

Since Γ is $\mathcal{R}$-polar and since $P_x\{\tilde{Z} \leq Z\} = 1$ by 5.4.A, we get (6.4). $\square$

Proof of Theorem 6.3. 1°. Let $\Lambda \in \mathcal{L}(\Gamma, \nu)$ and let $\Gamma_0 = \Gamma \setminus \Lambda$. The theorem will be proved if we show that $v = Q_{B_0}(u)$ is moderate for every closed subset B_0 of $O_0 = E' \setminus \Gamma_0$. Indeed, this implies $O_0 \subset O$ and therefore $\Gamma_0 \supset \Gamma$, $\Lambda = \emptyset$.

Let (Γ_1, ν_1) be the trace of v. By Lemma 6.3, it is sufficient to to prove that Γ_1 is $\mathcal{R}$-polar and ν_1 is finite.

2°. By 5.2.B, $v \leq u$, and by 5.2.A, all moderate sets for u are also moderate for v. Hence $\Gamma_1 \subset \Gamma$. By (N), $Q_B[Q_{B_0}](u) = 0$ for all $B \subset E' \setminus B_0$. Hence, $E' \setminus B_0$ is moderate for v and is contained in O_1. We conclude that $\Gamma_1 \subset B_0 \cap \Gamma$, which is a subset of $\mathcal{R}$-polar set Λ.

3°. Note that $B_0 \subset O \cup \Lambda$ does not contain explosion points of ν, and therefore $\nu(B_0) < \infty$. Since $Q_B(v) = 0$ for $B \cap B_0 = \emptyset$, the measure ν_1 vanishes on $E' \setminus B_0$. Since $v \leq u$, $\nu_1 \leq \nu$ on $O \subset O_1$. We have

$$\nu_1(O_1) = \nu_1(O_1 \cap B_0) = \nu_1[(O_1 \setminus O) \cap B_0] + \nu_1(O \cap B_0) \leq \nu(O \cap B_0) < \infty,$$

because $\nu_1(\Gamma \cap B_0) = 0$ by 5.1.B. $\square$

7. Maximal property of (Γ, ν)-solutions

7.1. Our objective is to prove:

Theorem 7.1. *The (Γ, ν)-solution corresponding to a normal pair (Γ, ν) dominates all solutions u with the same trace.*

This follows easily from a kind of mean value theorem which is also of independent interest:

Theorem 7.2. *Let u be a solution with the trace (Γ, ν). For every $D \in \mathcal{O}(\Gamma)$, there exists a positive function $Z^*(D)$ such that*

$$(7.1) \qquad\qquad Z^*(D) \leq A_\infty^\nu \quad a.s.$$

and

$$(7.2) \qquad\qquad u(x) = -\log P_x e^{-Z^*(D) - \langle u, X_D \rangle} \quad in\ D.$$

To prove Theorem 7.1, we apply Theorem 7.2 to domains $D_n = D(\Gamma, 1/n)$. Formulae (7.1) and (7.2) imply that

$$(7.3) \qquad\qquad u(x) \leq -\log P_x e^{-A^\nu} e^{-\langle u, X_{D_n} \rangle}$$

for every n. It follows from (5.14) and (5.17) that

$$u(x) \leq -\log P_x e^{-A^\nu} e^{-Z_\Gamma} \leq -\log P_x \{ Z_\Gamma = 0, e^{-A^\nu} \} \leq -\log \{ \mathcal{R} \cap \Gamma = \emptyset, e^{-A_\infty^\nu} \}$$

which proves Theorem 7.1.

7.2. Proof of Theorem 7.2. 1°. Denote by B the intersection of E' with the M-closure of D. Put $\tilde{D}_n = D \cap D_n$, where $D_n = D(B, 1/n)$. By the mean value property 2.1.D,

$$(7.4) \qquad\qquad u(x) = -\log P_x e^{-\langle u, X_{\tilde{D}_n} \rangle} \quad in\ \tilde{D}_n.$$

For every $x \in \tilde{D}_n$, the measure $X_{\tilde{D}_n}$ is concentrated, P_x-a.s., on $\bar{D} = \partial D \cup D$, and therefore

$$\langle u, X_{\tilde{D}_n} \rangle = \langle u_{\partial D}, X_{\tilde{D}_n} \rangle + \langle u_D, X_{\tilde{D}_n} \rangle \quad P_x\text{-a.s.}$$

Formula (7.2) will follow from (7.4) if we prove that, for every $x \in D$,

$$(7.5) \qquad\qquad \lim \langle u_{\partial D}, X_{\tilde{D}_n} \rangle = \langle u, X_D \rangle \quad P_x\text{-a.s.}$$

and

$$(7.6) \qquad\qquad \langle u_D, X_{\tilde{D}_n} \rangle \to Z^*(D) \quad P_x\text{-a.s.}$$

2°. Note that $\langle u_{\partial D}, X_{\tilde{D}_n} \rangle = \langle u, X_n \rangle$, where X_n is the restriction of the measure $X_{\tilde{D}_n}$ to ∂D. Therefore we get (7.5) if we prove that $X_n \uparrow X_D$ P_x-a.s. for $x \in D$. This has been already done at the end of the proof of Theorem 5.2.

3°. Denote by $\bar{X}_n$ the restriction of $X_{\tilde{D}_n}$ to D. By (1.12), (1.8) and 2.1.D,

$$P_x \{ e^{-\langle u_D, X_{\tilde{D}_{n+1}} \rangle} | \mathcal{F}_{\subset \tilde{D}_n} \} = P_{X_{\tilde{D}_n}} e^{-\langle u_D, X_{\tilde{D}_{n+1}} \rangle}$$

$$\geq P_{\bar{X}_n} e^{-\langle u_D, X_{\tilde{D}_{n+1}} \rangle} = e^{-\langle u, \bar{X}_n \rangle} = e^{-\langle u_D, X_{\tilde{D}_n} \rangle}.$$

Therefore $e^{-\langle u_D, X_{\tilde{D}_n} \rangle}$ is a bounded submartingale relative to P_x (cf. the proof of Theorem 3.1), and there exists $Z^*(D)$ subject to (7.6).

To get (7.1), we recall that, by Section 3.5, the restriction ν_B of ν to B is the trace of $Q_B(u)$. By Theorem 5.3 and 5.6.E, the corresponding functional A^{ν_B} satisfies the condition

$$A^{\nu_B} = \lim \langle u, X_{D_n} \rangle \quad \text{a.s.}$$

Lemma 3.1 in [9] implies that

$$\langle u_D, X_{\tilde{D}_n} \rangle \leq \langle u_D, X_{D_n} \rangle \leq \langle u, X_{D_n} \rangle.$$

Hence,

$$Z^*(D) = \lim \langle u_D, X_{\tilde{D}_n} \rangle \leq A^{\nu_B} \leq A^\nu \quad \text{a.s.} \qquad \square$$

8. On condition (ℕ)

8.1. In this section we introduce a concept of a normal point of the Martin boundary E' and we prove that condition (ℕ) holds if all points of E' are normal. In the second part of the section we establish tests of normality.

For every $c \in E'$, we put $V_\varepsilon(c) = \{x \in E : d(x, c) < \varepsilon\}$. More generally, $V_\varepsilon(B) = \{x \in E : d(x, B) < \varepsilon\}$ for every $B \subset E'$.

8.2. Admissible pairs and normal points. Let $c \in E'$ and let D be an open subset of E. We say that (c, D) is an *admissible pair* if there exist $\delta > 0$ and $\varepsilon > 0$ such that $V_\delta(c) \subset D$ and

$$(8.1) \qquad P_x\{X_D = 0\} \geq \varepsilon \qquad \text{for all} \;\; x \in V_\delta(c).$$

Note that, if (c, D) is admissible, then $(c, \tilde{D})$ is admissible for every $\tilde{D} \supset D$. Indeed, by (1.12),

$$P_x\{X_{\tilde{D}} = 0\} = P_x P_{X_D}\{X_{\tilde{D}} = 0\} \geq P_x\{X_D = 0\}.$$

A point $c \in E'$ is called *normal* if, for every $\varepsilon > 0$, there exists an admissible pair (c, D) such that $D \subset V_\varepsilon(c)$. Our first objective is to prove

Theorem 8.1. *Condition* (ℕ) *holds if all points of E' are normal.*

8.3. The proof needs some preparations. With every open subset D of E we associate a relatively open subset D^* of E' defined by the condition: $c \in D^*$ if $V_\varepsilon(c) \subset D$ for some $\varepsilon > 0$. Note that $\hat{D} = D \cup D^*$ is an open set in the exit space $\hat{E}$. Recall that to every function f on E there corresponds a random variable f^E defined by formula (2.3).

Lemma 8.1. *Suppose that τ is the first exit time from an open set $D \subset E$, φ is a positive bounded real-valued function on ∂D and*

$$f(x) = \Pi_x \varphi(\xi_\tau).$$

Then

$$(8.2) \qquad f^E = 0 \quad \text{a.s. on } \{\xi_{\zeta-} \in D^*\}.$$

Proof. It is sufficient to prove that

$$(8.3) \qquad \Pi_x\{\xi_{\zeta-} \in D_0^*, f^E\} = 0$$

for every $x \in E$ and every open set D_0 such that $d(D_0, E \setminus D) > 0$. Consider stopping times σ_n, τ_n defined as follows: $\tau_1 = \tau$;

$$\sigma_n = \inf\{t : t > \tau_n, \xi_t \in D_0\}$$

if $\xi_t \in D_0$ for some $t > \tau_n$; $\sigma_n = \zeta$ otherwise;

$$\tau_{n+1} = \inf\{t : t > \sigma_n, \xi_t \in E \setminus D\}$$

if $\xi_t \in E \setminus D$ for some $t > \sigma_n$, $\tau_{n+1} = \zeta$ otherwise. Put $A_n = \{\tau_1 < \sigma_1 < \cdots < \sigma_n < \tau_n = \zeta\}$. By the strong Markov property,

$$\Pi_x\{A_n, f^E\} = \Pi_x[\tau_1 < \sigma_1 < \cdots < \sigma_n, \Pi_{\xi_{\sigma_n}}\{\tau = \zeta, f^E\}].$$

By Lemma 2.1, $\Pi_y\{\tau = \zeta, f^E\} = \Pi_y\{\tau = \zeta, f^D\}$ is equal to 0 for all y, and therefore $\Pi_x\{A_n, f^E\} = 0$. Since the union of the A_n coincides with $\{\xi_{\zeta-} \in D_0^*\}$, this implies (8.3). $\qquad\square$

Lemma 8.2. *If*

$$(8.4) \qquad\qquad v(x) = -\log P_x\{X_D = 0\}$$

and if all points of D^* are normal, then for every $x \in E$,

$$(8.5) \qquad\qquad v^E = 0 \quad P_x\text{-a.s. on } \{\xi_{\zeta-} \in D^*\}.$$

For every closed set $B \subset D^*$, there exists $\delta > 0$ such that v is bounded in $V_\delta(B)$.

Proof. For each $c \in B$, there exists $\varepsilon(c) > 0$ such that v is bounded on $V[c] = V_{\varepsilon(c)}(c)$. The sets $\hat{V}[c]$ form an open cover of B. Let $\hat{V}[c_1], \ldots, \hat{V}[c_n]$ be a finite subcover and let D_1 be the union of the $V[c_i], i = 1, \ldots, n$. If $\delta > 0$ is sufficiently small, then $V_\delta(B) \subset D_1$, which proves the second part of the lemma.

To prove the first part, we apply Lemma 8.1 to $D_0 = V_{\delta/2}(B)$ and $\varphi = v$. By the Markov property (1.12),

$$v(x) = -\log P_x e^{-\langle v, X_{D_0}\rangle},$$

and, by Jensen's inequality, $v \leq f$, where $f(x) = P_x\langle v, X_{D_0}\rangle$. By (1.13), $f(x) = \Pi_x v(\xi_{\tau_0})$, where τ_0 is the first exit time from D_0. By Lemma 8.1, $f^E = 0$ a.s. on $\{\xi_{\zeta-} \in D_0^*\}$, which implies an analogous relation for v. Since B is an arbitrary closed subset of D^*, we get (8.5). $\qquad\square$

Lemma 8.3. *Let $\tilde{u} = Q_\Gamma(u)$. If all points of $E' \setminus \Gamma$ are normal, then, for every $\delta > 0$,*

$$(8.6) \qquad\qquad \tilde{u}^E = 0 \quad \text{a.s. on } \{\xi_{\zeta-} \notin V_\delta(\Gamma)^*\}.$$

For every closed set B disjoint from Γ, there exists $\delta > 0$ such that $\tilde{u}$ is bounded in $V_\delta(B)$.

Proof. If $B \subset E' \setminus \Gamma$, then there is a $D \in \mathcal{O}(\Gamma)$ such that $B \subset D^*$. Since $1_{X_D=0} \leq e^{-\langle u, X_D\rangle}$, we have $\tilde{u} \leq V_D(u) \leq v$, where v is given by (8.4). Therefore Lemma 8.3 follows from Lemma 8.2. $\qquad\square$

8.3. Proof of Theorem 8.1. Put $\tilde{u} = Q_\Gamma(u)$ and $v = Q_B(\tilde{u})$. We need to prove that $v = 0$. By Theorem 2.1, it is sufficient to show that

$$(8.7) \qquad\qquad v^E = 0 \quad \text{a.s.}$$

and that v is bounded. By Lemma 8.3, (8.6) holds for sufficiently small $\delta > 0$. By 5.2.B, $v \leq \tilde{u}$ and therefore $v^E = 0$ a.s. on $\{\xi_{\zeta-} \notin V_\delta(\Gamma)^*$. By the same lemma, if δ is sufficiently small, then $v^E = 0$ a.s. on $\{\xi_{\zeta-} \notin V_\delta(B)^*\}$. Condition (8.7) is satisfied because $V_\delta(\Gamma)^*$ and $V_\delta(B)^*$ are disjoint if $\delta < d(B, \Gamma)/2$.

Denote by B_1 the set of all $x \in E'$ such that $d(x, B) \leq d(B, \Gamma)/2$, and let Γ_1 be the closure of $E' \setminus B_1$. Since $B_1 \cap \Gamma = \emptyset$, by Lemma 8.3, $\tilde{u}$ is bounded in $V_\delta(B_1)$

for sufficiently small δ and the same is true for v. Since $B \cap \Gamma_1 = \emptyset$, v is bounded in $V_\delta(\Gamma_1)$ for sufficiently small δ. Since $B_1 \cup \Gamma_1 = E'$, v is bounded in $V_\delta(E')$. The mean value property 2.1.D and Jensen's inequality imply that v is bounded in E. $\square$

8.4. Barriers. A *barrier* in an open set $D \subset E$ is a positive function u such that

$$(8.8) \qquad\qquad Lu \leq u^\alpha \quad \text{on } D$$

and, for all $x \in D$,

$$(8.9) \qquad\qquad u^D = \infty \quad \Pi_x\text{-a.s. on } \{\tau < \zeta\},$$

where τ is the first exit time from D. Condition (8.9) holds, in particular, if

$$(8.10) \qquad\qquad u = \infty \quad \text{on } \partial D.$$

Theorem 8.2. *If u is a barrier in D, then*

$$(8.11) \qquad v(x) = -\log P_x\{X_D = 0\} \leq u(x) \qquad \text{for all } x \in D.$$

Proof. Note that $v_k \uparrow v$, where

$$v_k(x) = -\log P_x e^{-k\langle 1, X_D \rangle}.$$

Therefore it is sufficient to prove that $v_k \leq u$ for all k. By Jensen's inequality, $v_k \leq kh$, where $h(x) = P_x\langle 1, X_D \rangle = \Pi_x\{\tau < \zeta\}$ (cf. the proof of Lemma 8.2). For every $x \in D$, by Lemma 2.1,

$$h^D = 1 \quad \text{a.s. on } \{\tau < \zeta\}, \quad h^D = 0 \quad \text{a.s. on } \{\tau = \zeta\}$$

and therefore

$$(8.12) \qquad v_k^D \leq 1 \quad \text{a.s. on } \{\tau < \zeta\}, \quad v_k^D = 0 \quad \text{a.s. on } \{\tau = \zeta\}.$$

On the other hand, by (8.9),

$$(8.13) \qquad u^D = \infty \quad \text{a.s. on } \{\tau < \zeta\}, \quad u^D \geq 0 \quad \text{a.s. on } \{\tau = \zeta\}.$$

By (8.12) and (8.13), $(v_k - u)^D \leq 0$ Π_x-a.s. Since $v_k - u \leq k$ and $Lu - u^\alpha \leq 0 = Lv_k - v_k^\alpha$, $v_k \leq u$ in D by Theorem 2.1. $\square$

Corollary. *A pair (c, D), where $c \in D^*$, is admissible if there exists a barrier u in D that is bounded in $V_\delta(c)$ for some $\delta > 0$.*

8.5. Bounded domains in $\mathbb{R}^d$. Now we investigate (L, α)-superdiffusion under the following assumptions:

   8.5.A. E is a bounded domain in $\mathbb{R}^d$.

   8.5.B. The coefficients of the operator

$$L = \sum_{i,j=1}^{d} a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b_i(x) \frac{\partial}{\partial x_i}$$

satisfy the following conditions: there exist constants K_1, K_2 such that

$$0 < \sum_{i,j=1}^{d} a_{ij}(x) \lambda_i \lambda_j \leq K_1 \sum_{i=1}^{d} \lambda_i^2$$

and

$$\sum_{i=1}^{d} b_i(x)^2 \leq K_2$$

for all $x \in D$ and all $\lambda_1, \ldots, \lambda_d \in \mathbb{R}$.

We use the notation $d_0(x, y)$ for the Euclidean distance between x and y. We denote by ∂E the boundary of E in the Euclidean topology of $\mathbb{R}^d$, and we put $U_\varepsilon(B) = \{x \in E : d_0(x, B) < \varepsilon\}$ for an arbitrary $B \subset \partial E$. We say that points $c \in E'$ and $b \in \partial E$ are *associated* and we write $c \hookrightarrow b$ if there exists a sequence $x_n \in E$ such that $d(x_n, c) \to 0$ and $d_0(x_n, b) \to 0$. Note that the set $C_b = \{c \in E' : c \hookrightarrow b\}$ is M-closed, and the set $B_c = \{b \in \partial E : c \hookrightarrow b\}$ is closed in the Euclidean topology.

Theorem 8.3. *Under conditions 8.5.A, B, the pair $(c, U_\varepsilon(B_c)))$ is admissible for every $c \in E'$.*

The proof of Theorem 8.3 is based on the following lemmas.

Lemma 8.4. *For every $\varepsilon > 0$ and every $b \in \partial E$, there exists a constant λ such that*

$$(8.14) \qquad\qquad v_\varepsilon(x) = \lambda[\varepsilon^2 - d_0(x, b)^2]^{-2/(\alpha - 1)}$$

is a barrier in $U_\varepsilon(b)$.

Proof. Clearly, v_ε satisfies condition (8.10). A direct computation (see [3, p.102]) shows that $Lv_\varepsilon - v_\varepsilon^\alpha \leq 0$ for sufficiently large λ. $\qquad\square$

Lemma 8.5. *For every $\varepsilon > 0$, there exists $\delta > 0$ such that $V_\delta(c) \subset U_\varepsilon(B_c)$.*

Proof. If this is false, then there exist $\varepsilon > 0$ and $x_n \in E$ such that $d(x_n, c) < 1/n$ but $d_0(x_n, B_c) > \varepsilon$. Choose a subsequence x_{n_k} which converges in the Euclidean topology. Clearly, its limit b belongs to B_c, in contradiction to the inequality $d_0(x_{n_k}, B_c) > \varepsilon$. $\qquad\square$

Proof of Theorem 8.3. For every $b \in B_c$, $U_\varepsilon(b) \subset U_\varepsilon(B_c)$. Hence

$$u(x) = -\log P_x\{X_{U_\varepsilon(B_c)} = 0\}$$

is dominated by

$$v^b(x) = -\log P_x\{X_{U_\varepsilon(b)} = 0\}.$$

By Theorem 8.2 and Lemma 8.4, v^b does not exceed the barrier v_ε defined by (8.14). We conclude that u is bounded on $U_{\varepsilon/2}(b)$ for every $b \in B_c$. Since B_c is compact, it contains a finite subset $\{b_1, \ldots, b_n\}$ such that, for every $b \in B_c$, $\min_i d_0(b, b_i) < \varepsilon/4$. Therefore $U_{\varepsilon/4}(B_c) \subset U_{\varepsilon/2}(b_1) \cup \cdots \cup U_{\varepsilon/2}(b_n)$. By Lemma 8.5, there exists $\delta > 0$ such that $V_\delta(c) \subset U_{\varepsilon/4}(B_c)$. Hence u is bounded on $V_\delta(c)$. $\qquad\square$

8.6.

Theorem 8.4. *A point $c \in E'$ is normal if $B_c \cap B_{c'}' = \emptyset$ for all $c' \neq c$.*

Proof. It follows from Theorem 8.3 that c is normal if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that $U_\delta(B_c) \subset V_\varepsilon(c)$. Suppose that this condition is not satisfied. Then there exist $\varepsilon > 0$ and a sequence $x_n \in E$ such that $d_0(x_n, B_c) < 1/n$ and $d(x_n, c) \geq \varepsilon$. Choose a subsequence x_{n_k} which converges relative to both metrics d_0 and d. Denote by b and c' its Euclidean limit and M-limit. Clearly, $b \in B_c \cap B_{c'}$ and $d(c', c) \geq \varepsilon$. $\qquad\square$

Theorem 8.5. *A point $c \in E'$ is normal if B_c consists of a single point b and if c is an isolated point in C_b.*

First we establish two lemmas.

Lemma 8.6. *For every $\varepsilon > 0$, there exists $\delta > 0$ such that $U_\delta(b) \subset V_\varepsilon(C_b)$.*

The proof is similar to the proof of Lemma 8.5.

Lemma 8.7. *Suppose D_1, D_2 are two disjoint open subsets of E and $D = D_1 \cup D_2$. Then, for every $x \in D_1$, $P_x\{X_{D_1} = X_D\} = 1$.*

Proof. By Lemma 3.1 in [9, p.1978], $X_D(\Gamma) \geq X_{D_1}(\Gamma)$ P_x-a.s. for every $\Gamma \subset D$. By (1.13), $P_x X_D(\Gamma) = \Pi_x\{\xi_\tau \in \Gamma\}$ and $P_x X_{D_1}(\Gamma) = \Pi_x\{\xi_{\tau_1} \in \Gamma\}$, where τ, τ_1 are the first exit times from D and from D_1. If $x \in D_1$, then $\tau = \tau_1$ Π_x-a.s., and therefore $P_x X_D(\Gamma) = P_x X_{D_1}(\Gamma)$. $\square$

Proof of Theorem 8.5. It is sufficient to prove that the pair $(c, V_\varepsilon(c))$ is admissible for all sufficiently small ε. There exists $\beta > 0$ such that $d(c, c') > 2\beta$ for all $c' \in C(b), c' \neq c$. Suppose that $\varepsilon < \beta$. Then $V_\varepsilon(C_b) = V_\varepsilon(c) \cup \tilde{V}$, where $\tilde{V}$ is open and disjoint from $V_\varepsilon(c)$. By Lemma 8.6, $U_\delta(b) \subset V_\varepsilon(C_b)$ for some δ. By Theorem 8.3, $(c, U_\delta(B_c))$ is admissible. Hence $(c, U_\delta(b))$ and $(c, V_\varepsilon(C_b))$ are also admissible. It remains to note that, by Lemma 8.7, $X_{V_\varepsilon(C_b)} = X_{V_\varepsilon(c)}$ P_x-a.s. for all $x \in V_\varepsilon(c)$. $\square$

References

1. M. Brelot, *On Martin boundaries, Unpublished lectures*, Hiroshima University, 1962; Russian translation in Matematika **9:5** (1965),136-155.
2. E. B. Dynkin, *Markov Processes*, Springer-Verlag, Berlin, Heidelberg, 1965. MR **33:**1887
3. _____, *A probabilistic approach to one class of nonlinear differential equations*, Probab. Th. Rel. Fields **89** (1991), 89-115. MR **92d:**35090
4. _____, *Superdiffusions and parabolic nonlinear differential equations*, Ann. Probab. **20** (1992), 942-962. MR **93d:**60124
5. _____, *Superprocesses and partial differential equations*, Ann. Probab. **21** (1993), 1185-1262. MR **94j:**60156
6. _____, *A probabilistic approach to a nonlinear differential equation on a Riemannian manifold*, Teoriya Veroyatnostei i ee Primeneniya **42** (1997), 336–341; English transl., to appear in Theory Probab. Appl. 42 (1997). CMP 98:02
7. E. B. Dynkin and S. E. Kuznetsov, *Superdiffusions and removable singularities for quasilinear partial differential equations*, Comm. Pure & Appl. Math **49** (1996), 125-176. MR **97m:**60114
8. _____, *Solutions of $Lu = u^\alpha$ dominated by L-harmonic functions*, Journale d'Analyse Math. **68** (1996), 15-37. MR **97f:**35048
9. _____, *Linear additive functionals of superdiffusions and related nonlinear P.D.E.*, Trans. Amer. Math. Soc. **348** (1996), 1959-1987. MR **97d:**60135
10. _____, *Nonlinear parabolic P.D.E. and additive functionals of superdiffusions*, Ann. Probab. **25** (1997), 662–701. CMP 97:08
11. _____, *Natural linear additive functionals of superprocesses*, Ann. Probab. **25** (1997), 640–661. CMP 97:08
12. _____, *Trace on the boundary for solutions of nonlinear differential equations*, Trans. Amer. Math. Soc. **350** (1998), 4499–4519.
13. J.F. Le Gall, *Solutions positives de $\Delta u = u^2$ dans le disque unité*, C.R. Acad. Sci. Paris, Série I **317** (1993), 873-878. MR **94h:**35059
14. _____, *A probabilistic Poisson representation for positive solutions of $\Delta u = u^2$ in a planar domain*, Comm. Pure Appl. Math. **50** (1997), 69–103. MR **88e:**60144
15. M. Marcus and L. Véron, *Trace au bord des solutions positives d'équations elliptiques non linéaires*, C.R. Acad.Sci Paris ser **I 321** (1995), 179-184. MR **96f:**35045
16. _____, *Trace au bord des solutions positives d'équations elliptiques non linéaires. Résultats d'existence et d'unicité*, C.R. Acad.Sci Paris ser **I 323** (1996), 603-608. MR **97f:**35012

17. _____ , *The boundary trace of positive solutions of semilinear elliptic equations* I: *The subcritical case.* Arch. Rational Mech. Anal., 1998, to appear.

17a. M. Marcus and L. Véron, *The boundary trace of positive solutions of semilinear elliptic equations* II: *The supercritical case.* Preprint, 1997.

18. V. G. Maz'ya, *Beurling's theorem on a minimum principle for positive harmonic functions*, [First published (in Russian) in Zapiski Nauchnykh Seminarov Leningradskogo Otdeleniya Mat. Inst. im. V. A. Steklova 30 (1972), 76-90], J.Soviet Math. **4**, 367-379. MR **48:**8821

19. K.Yosida, *The fundamental solution of the parabolic equation in a Riemannian space*, Osaka Math. J. **5:1** (1953), 65-74. MR **15:**36b

DEPARTMENT OF MATHEMATICS, WHITE HALL, CORNELL UNIVERSITY, ITHACA, NEW YORK 14853-7901

E-mail address: `ebd1@cornell.edu`

CENTRAL ECON.-MATH. INSTITUTE, RUSSIAN ACADEMY OF SCIENCES, 32 KRASIKOWA, MOSCOW 117418, RUSSIA

Current address: Department of Mathematics, University of Colorado at Boulder, Boulder, Colorado 80309-0395

E-mail address: `sk47@cornell.edu`